On mode conversion and wave reflection in magnetic Ap stars

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ABSTRACT
We investigate the effect of a strong large-scale magnetic field on the reflection of high-frequency acoustic modes in rapidly oscillating Ap stars. To that end, we consider a toy model composed of an isothermal atmosphere matched on to a polytropic interior and determine the numerical solution to the set of ideal magnetohydrodynamic equations in a local plane-parallel approximation with constant gravity. Using the numerical solution in combination with approximate analytical solutions that are valid in the limits where the magnetic and acoustic components are decoupled, we calculate the relative fraction of energy flux that is carried away in each oscillation cycle by running acoustic waves in the atmosphere and running magnetic waves in the interior. For oscillation frequencies above the acoustic cut-off, we show that most energy losses associated with the presence of running waves occur in regions where the magnetic field is close to vertical. Moreover, by considering the depth dependence of the energy associated with the magnetic component of the wave in the atmosphere we show that a fraction of the wave energy is kept in the oscillation every cycle. For frequencies above the acoustic cut-off frequency, such energy is concentrated in regions where the magnetic field is significantly inclined in relation to the local vertical. Even though our calculations were aimed at studying oscillations with frequencies above the acoustic cut-off frequency, based on our results we discuss what results may be expected for oscillations of lower frequency.

Key words: stars: magnetic fields – stars: oscillations – stars: variables: other.

1 INTRODUCTION
A number of rapidly oscillating Ap stars, including the two prototypes $\alpha$ Cir (Kurtz et al. 1994) and HR1217 (Kurtz et al. 2005), are known to pulsate with frequencies that exceed the acoustic cut-off frequency expected from appropriate stellar models. This fact has been a matter of long debate over the years (e.g. Shibahashi & Saio 1985; Audard et al. 1998; Cunha 1998; Gautschy, Saio & Harzenmoser 1998). Generally, it has been suggested that the apparent dilemma posed by the observation of oscillations of such high frequency in roAp stars results from the use of a temperature–optical depth ($T$–$\tau$) relation that is inadequate for these peculiar stars. However, except for the case in which the $T$–$\tau$ relation was modified to simulate the presence of a chromosphere, for which no observational evidence exists, models with modified $T$–$\tau$ relations considered in the above-mentioned works have not been able to fully resolve the apparent discrepancy.

Even though the oscillations observed in roAp stars have essentially an acoustic nature, it is nowadays well established that the latter acquire a magnetoacoustic nature in the outer layers of the star, where the magnetic and gas pressure become comparable. A number of studies investigating on the effect of the magnetic field on the properties of the oscillations, such as eigenfrequencies and eigenfunctions, were carried out by different authors (e.g. Dziembowski & Goode 1996; Bigot et al. 2000; Cunha & Gough 2000; Saio & Gautschy 2004; Cunha 2006). Mode conversion – that is, the process by which the acoustic wave in the interior passes part of its energy to waves of a different nature through the coupling with the magnetic field in the region where the gas and magnetic pressures are comparable – was considered in all the above mentioned works. In some of them, the waves were artificially reflected at the surface of the star. In those cases, only the energy passed on to running magnetic waves in the interior was considered. In the studies of Dziembowski & Goode (1996) and Cunha (2006), on the other hand, the authors did not make the former assumption, and as a result, part of the energy associated with the acoustic wave in the interior was allowed to be converted, through the coupling with the magnetic field, into running acoustic waves in the high atmosphere, as well as into running magnetic waves in the interior.

While incorporating the process of mode conversion, the works mentioned above aimed mostly at studying the overall effect of the presence of the magnetic field on the eigenfrequencies and eigenfunctions of the oscillations. Thus, in no case the impact that such a process might have on the issue of mode reflection at the surface of roAp stars was properly analysed. This issue, which is directly

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related to the long-debated problem of the cut-off frequency in roAp stars discussed at the beginning of this section, is studied in this work in a toy model composed of an isothermal atmosphere matched on to an index 3 polytropic interior. We note that the phenomena of mode conversion are known to take place in other classes of pulsating stars. In particular, in recent years a number of studies related to mode conversion were carried out also in the context of solar pulsations in regions of strong magnetic field (e.g. McDougall & Hood 2007, and references therein).

2 THE MAGNETIC BOUNDARY LAYER

2.1 Governing equations and general assumptions

In this work, we are particularly concerned with pulsations in Ap stars. Observations indicate that Ap stars have intense, large-scale, magnetic fields, often with predominant dipolar structure (e.g. Wade et al. 2000; Hubrig et al. 2004, 2005). Accordingly, in this work we will consider that our stellar model is permeated by a dipolar magnetic field. Since this is a force-free field, it does not influence the equilibrium structure of the star, which therefore is governed by the system of equations

\[
\frac{\partial \rho_0}{\partial t} = 0, \quad (1)
\]

\[
\frac{\partial \rho_0}{\partial t} = 0, \quad (2)
\]

\[
\nabla \cdot \mathbf{v}_0 = \rho_0 \mathbf{g}_0, \quad (3)
\]

where \(\rho_0\) is the density, \(p_0\) is the pressure, \(\mathbf{g}_0\) is the gravitational field, and the subscript 0 refers to the equilibrium quantities. In the above equations, \(\nabla\) stands for partial derivatives and \(t\) is the time.

In the limit of perfect conductivity, adiabatic motions are governed by the set of ideal magnetohydrodynamic (MHD) equations (see e.g. Priest 1982). Considering small deviations from the equilibrium state and then linearizing the MHD equations by neglecting squares and products of the small quantities (that will be denoted \(\beta\) and \(\gamma\)), we neglect the Eulerian perturbation to the gravitational potential (Cowling approximation), we arrive at the following system of equations:

\[
\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho_0} \nabla p_1 + \frac{\rho_0}{\mu_0} \nabla \rho_0 + \frac{1}{\mu_0} (\nabla \times \mathbf{B}_1) \times \mathbf{B}_0, \quad (4)
\]

\[
\frac{\partial p_1}{\partial t} = -\mathbf{v} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \mathbf{v}, \quad (5)
\]

\[
\frac{\partial \rho_1}{\partial t} = -\mathbf{v} \cdot \nabla \rho_0 - \rho_0 \nabla \cdot \mathbf{v}, \quad (6)
\]

\[
\frac{\partial \mathbf{B}_1}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}_0), \quad (7)
\]

\[
\nabla \cdot \mathbf{B}_1 = 0, \quad (8)
\]

where \(\mathbf{v}\) is the pulsation velocity, \(\gamma\) is the first adiabatic exponent, \(\mu\) is the magnetic permeability, and \(\mathbf{B}\) is the magnetic field. The perturbed Lorentz force in equation (4) expresses the magnetic field influence on the oscillations.

In our study, we will deal only with the outer layers of the star. We will consider the following three different regions (cf. Fig. 1).

(i) A region in the atmosphere where the magnetic pressure dominates the gas pressure (\(\beta^-\) region).

(ii) A region where the magnetic and the gas pressure are of the same order of magnitude.

(iii) A region where the magnetic pressure is negligible when compared with the gas pressure (\(\beta^+\) region).

Here, \(\beta\) is defined as the ratio between the gas pressure, \(p_g\), and the magnetic pressure, \(p_m = B^2 / 2\mu_0\). The region \(\beta^-\) corresponds to the layers where \(\beta \ll 1\), while the region \(\beta^+\) corresponds to the layers where \(\beta \gg 1\).

We will refer to the two outer regions as the magnetic boundary layer. We note that since in the interior of the star the gradient of the gas pressure is much greater than the Lorentz stresses, the dynamics there is effectively field-free. Thus, the magnetic field influences the global oscillations only through its direct contribution to the restoring force in the magnetic boundary layer.

In this study, we will consider perturbations to the equilibrium state of the model associated with high-order, low-degree, axisymmetric pulsations only. These are the pulsations that are typically observed in roAp stars. Moreover, we will not consider pure incompressible perturbations (even though in the \(\beta^+\) region the decoupled magnetic component becomes almost incompressible).

Since the magnetic field is assumed to have a dipolar configuration and the pulsations are axisymmetric, the perturbations depend on co-latitude \(\theta\) but not on the axial angle \(\phi\).

Under these conditions, the system of equations (4)–(8) admits solutions of the form

\[
\mathbf{\xi}(r, \theta, t) = \mathbf{\xi}(r, \theta) e^{i\omega \tau}, \quad (9)
\]

where \(\mathbf{\xi}\) is the displacement vector defined by \(v = \frac{\partial \mathbf{\xi}}{\partial \tau}\), \(\omega\) is the oscillation frequency and \(r\) is the radial coordinate, in the spherical coordinate system \((r, \theta, \phi)\).

Since the boundary layer is thin and the magnetic field varies only on large scales, we follow Dziembowski & Goode (1996) and Cunha & Gough (2000) and solve our problem locally by adopting, at each latitude, a plane-parallel approximation, and assuming a locally uniform field.

At each given latitude, our Cartesian system is defined such that the local vertical coordinate \(z\) increases outwardly and is zero at the surface of the star. The coordinates in the horizontal plane, \(x\) and \(y\), are chosen such that the \(x\)-axis is parallel to the horizontal component of the magnetic field. Hence, \(\mathbf{B}_0 = (B_y, 0, B_z)\).

In accordance with our approximation of a locally uniform magnetic field, we disregard the derivatives of \(B_x\) and \(B_z\) in
equations (4)–(8). Since the oscillations considered are axisymmetric, \( \partial \xi / \partial y = 0 \). Moreover, given that the observed modes are of low degree, we expect that \( \xi \) will vary much more rapidly in the \( z \) direction than in the \( x \) direction. Hence, we also approximate \( \partial \xi / \partial x \approx 0 \). The latter approximation is exact for pulsations that in the absence of the magnetic field would be spherically symmetric (i.e. with degree \( l \) = 0).

The presence of a magnetic field, and the consequent Lorentz force that is generated, introduces horizontal motion in the magnetic boundary layer even in the case of modes that in the absence of a magnetic field would be radial. We can decompose the displacement into its vertical component and its horizontal component defined, respectively, by \( \xi_\parallel = \xi \cdot \hat{e}_\parallel \) and \( \xi_\perp = \xi \cdot \hat{e}_\perp \), where \( e_\parallel \) and \( e_\perp \) are unit vectors in the \( z \)- and \( x \)-directions, respectively. The fact that the horizontal displacement is along \( \hat{e}_\parallel \) is a consequence of the axisymmetry of the problem.

Using equations (5)–(7) to eliminate \( p_1, \rho_1 \) and \( B_1 \) from equation (4), neglecting the derivatives of \( B_\parallel \) and \( B_\perp \), and simplifying, we obtain

\[
-w^2 \rho_0 \xi_\parallel = -\frac{B_\parallel}{\mu} [(B_\parallel \cdot \nabla)(\nabla \cdot \xi)] + \frac{1}{\mu} (B_\parallel \cdot \nabla)^2 \xi_\parallel,
\]

\[
-w^2 \rho_0 \xi_\perp = \frac{\partial W}{\partial z} - \frac{B_\parallel}{\mu} [(B_\parallel \cdot \nabla)(\nabla \cdot \xi)] + \frac{1}{\mu} (B_\parallel \cdot \nabla)^2 \xi_\perp - \frac{1}{\rho_0} \frac{\partial p_0}{\partial z} \nabla \cdot (\rho_0 \xi),
\]

where

\[
W = \xi \cdot \nabla p_0 + \left( \gamma \frac{p_0 + B_\parallel^2}{\mu} \right) \nabla \cdot \xi - \frac{1}{\mu} (B_\parallel \cdot \nabla)(B_\parallel \cdot \xi).
\]

2.2 Dimensionless variables

When solving our problem, we will make the system of equations (10) and (11) dimensionless by defining the new variables

\[
\eta = -z/R, \quad \sigma = \omega / \omega_0, \quad p = p_0 / \rho_0, \quad \rho = \rho_0 / \rho_0, \quad \xi_\parallel = \xi_\parallel / R, \quad \xi_\perp = \xi_\perp / R, \quad b_\parallel = B_\parallel / (\mu \rho_0 \omega_0^2 R^2)^{1/2}, \quad C_2 = \rho_0 / (\mu \rho_0 \omega_0^2 R^2),
\]

where \( \rho_0 \) is the pressure at the stellar surface, \( \rho_0 \) the density at the stellar surface and \( \omega_0 = \sqrt{GM/R^2} \), where \( M \) and \( R \) are, respectively, the mass and radius of the star, and \( G \) is the gravitational constant. Following the same notation, the dimensionless magnetic field vector becomes \( \hat{b}_\parallel = (b_x, 0, b_z) \) and the total dimensionless displacement becomes \( \hat{\xi} = \xi_\parallel \hat{e}_\parallel + \xi_\perp \hat{e}_\perp \). Moreover, we define the dimensionless sound speed as \( c = (C_2 \gamma p/\rho)^{1/2} \).

Combining these new variables with the system of equations (10) and (11) and simplifying, we obtain

\[
-b_\parallel b_\parallel \xi_\parallel'' + b_\perp^2 \xi_\perp'' + \sigma^2 \rho \xi_\perp = 0,
\]

\[
(\gamma C_2 \rho + b_\parallel^2) \xi_\parallel'' + (\gamma C_2 \rho') \xi_\perp'' + \sigma^2 \rho \xi_\perp - b_\parallel b_\parallel \xi_\parallel'' - b_\perp b_\perp \xi_\perp'' = 0,
\]

where the prime ' stands for derivatives in order to the new independent variable \( \eta \).

2.3 Magnetic coordinate system

When the system of equations (13) and (14) is studied in the extreme case of no gas pressure it is found that the displacement is perpendicular to the magnetic field. This is a direct consequence of the Lorentz force being the only restoring force playing a role in that extreme case. With this in mind we will define a second coordinate system with axes directed along and perpendicular to the magnetic field (see Fig. 2). To change from the first to the second coordinate system, we use the relations

\[
\begin{cases}
\xi_\parallel = \xi_\parallel \cos \alpha - \xi_\perp \sin \alpha \\
\xi_\perp = \xi_\parallel \sin \alpha + \xi_\perp \cos \alpha,
\end{cases}
\]

where \( \alpha \) is the inclination angle of the magnetic field in relation to the horizontal direction and \( \xi_\parallel \) and \( \xi_\perp \) are the dimensionless components of the displacement along and perpendicular to the magnetic field direction, respectively. This coordinate system will be useful when dealing with the problem in the \( \beta^- \) region. On the other hand, it will be useful to use the \((x, z)\) coordinate system in the \( \beta^+ \) region, since in the region where the gas pressure dominates the magnetic pressure the vertical component of the displacement is essentially an acoustic wave (e.g. Campbell & Papaloizou 1986; Cunha & Gough 2000).

3 STELLAR MODEL

The main goal of this work is to understand how high-frequency magnetoacoustic pulsations are reflected in the outer layers of Ap stars, and what fraction of the pulsation energy is lost through running waves. Thus, we start by considering a simple model that still contains the main physical ingredients necessary to study this problem, namely a plane-parallel envelope under constant gravitational acceleration, composed of an isothermal atmosphere matched at the surface on to a polytropic interior of index 3. We hope to refine our analysis using a more sophisticated stellar model, in future work.

All results presented are for a model of mass \( M = 2.0 M_\odot \) and radius \( R = 2.0 R_\odot \), where \( R_\odot \) and \( M_\odot \) are the radius and mass of the sun, respectively. The temperature of the isothermal atmosphere
4 DECOUPLING REGIONS

4.1 $\beta^+\parallel$ region

As shown by Roberts & Soward (1983), deep enough in the star the magnetoacoustic wave decouples into its fast acoustic and slow magnetic components, and it is possible to find an approximate solution for the horizontal component of the dimensionless displacement $\epsilon_\parallel$, by applying appropriate standard asymptotic theory valid for high-frequency modes. Within the approximations considered in this work, the horizontal component of the displacement in the $\beta^+\parallel$ region is entirely magnetic and has the form

$$
\epsilon_\parallel \approx \rho^{-1/4} D \exp \left[ -i \int_0^\eta \left( \frac{\rho \sigma^2}{b_c^2} \right)^{1/2} d\eta \right] + \rho^{-1/4} E \exp \left[ i \int_0^\eta \left( \frac{\rho \sigma^2}{b_c^2} \right)^{1/2} d\eta \right], \quad (22)
$$

where $D$ and $E$ are depth-independent amplitudes. Following the same authors we assume that this component of the displacement is an inwardly propagating wave and set $E = 0$. This is justified by the fact that the latter describes a magnetic wave that is expected to dissipate while propagating inwardly, due to the rapid increase of its wavenumber with depth.

4.2 $\beta^-\perp$ region

If the oscillation frequency is sufficiently high, the wave will propagate in the atmosphere. In that case, we expect the displacement to decouple into a slow acoustic and a fast magnetic component, as a result of the difference in the characteristic magnetic and acoustic wavelengths in this region. Thus, under these conditions, the total displacement may be written as the sum of fast (subscript $f$) and slow (subscript $s$) oscillatory components as

$$
\epsilon = \epsilon_f + \epsilon_s. \quad (23)
$$

In the $\beta^-\perp$ region, the perturbed Lorentz force dominates the restoring force for perturbations that are perpendicular to the unperturbed magnetic field. Hence, the component of the displacement in the direction perpendicular to the unperturbed magnetic field will be associated essentially with a compressible magnetic wave. However, to first order the Lorentz force has no effect for perturbations that are perpendicular to the unperturbed magnetic field. Thus, the dimensionless displacement along that direction will be associated with a wave that is essentially acoustic. We thus expect to have

$$
\epsilon_{\parallel s} \gg \epsilon_{\perp s}, \quad (24)
$$

and

$$
\epsilon_{\perp f} \gg \epsilon_{\parallel f}. \quad (25)
$$

Rewriting the system of equations (13) and (14) in the coordinate system defined by equations (15), we find

$$
\gamma C_s \rho \left( \epsilon_{\perp s}^* + \epsilon_{\parallel f} \frac{\sin \alpha}{\cos \alpha} \right) + \gamma C_s \rho' \left( \epsilon_{\perp s} + \epsilon_{\parallel f} \frac{\sin \alpha}{\cos \alpha} \right) + \frac{\sigma^2 \rho}{\sin \alpha \cos \alpha} \epsilon_{\parallel f} = 0, \quad (26)
$$

$$
\frac{b_c}{\cos \alpha} \epsilon_{\perp s}^* + \sigma^2 \rho (\epsilon_{\parallel f} \cos \alpha - \epsilon_{\perp s} \sin \alpha) = 0. \quad (27)
$$

Since the coefficients multiplying the displacement in equations (26) and (27) are not wave like functions, these equations must be
satisfied also by the fast and slow components of the displacement separately. Neglecting $\varepsilon_{1s}$ when compared with $\varepsilon_{s}$ and $\varepsilon_{f}$ when compared with $\varepsilon_{f,1}$, we find that the equations governing the slow and fast components of the displacement are, within the referred approximations,

$$\varepsilon''_{1s} + \frac{p'}{p} \varepsilon'_{1s} + \frac{\sigma^2 \rho}{\gamma C_2 p \sin \alpha} \varepsilon_{1s} = 0$$

and

$$\varepsilon''_{s} + \frac{p'}{(p + B^2)} \varepsilon'_{s} + \frac{\sigma^2 \rho}{\gamma C_2 \cos^2 \alpha (p + B^2)} \varepsilon_{s} = 0,$$

respectively, where

$$\beta^2 = \frac{|\mathbf{b}_0|^2}{\gamma C_2 \cos^2 \alpha}.$$  

This decoupling of the magnetoacoustic wave into acoustic and magnetic components in the $\beta^-$ region will be verified numerically in Section 5. Equations (28) and (29) may be written in the form of a standard wave equation. To that end, we define the new variable

$$\Xi_{1s} = p^{1/2} \varepsilon_{1s},$$

and introduce it in equation (28) to find

$$\Xi''_{1s} + k^2_{1s} \Xi_{1s} = 0,$$

where

$$k^2_{1s} = \left[ \left( \frac{p'}{2p} \right)^2 - \frac{1}{2} \frac{p''}{p} + \frac{\sigma^2 \rho}{\gamma C_2 p \sin \alpha} \right].$$

For equation (29), we can use a similar variable, defined by

$$\Xi_{s} = (p + B^2)^{1/2} \varepsilon_{s,1},$$

to find

$$\Xi''_{s} + k^2_{s} \Xi_{s} = 0,$$

where

$$k^2_{s} = \left[ \left( \frac{p'}{2(p + B^2)} \right)^2 - \frac{1}{2} \frac{p''}{p} + \frac{\sigma^2 \rho}{\gamma C_2 (p + B^2) \cos^2 \alpha} \right].$$

Equations (28) and (29) were derived under the assumption that two decoupled wave components with significantly different characteristic wavelengths can be identified in the $\beta^-$ region. That condition is satisfied when $k^2_{1s}$ and $k^2_{s}$ are both positives, $k^2_{1s}$ is positive for oscillations with frequencies larger than $\sigma_{c}$, where

$$\sigma_c = \frac{c}{2H} \sin \alpha.$$  

Except for the dependence on the angle $\alpha$, this expression is similar to the cut-off frequency for acoustic waves in an isothermal atmosphere with no magnetic field (hereafter named acoustic cut-off frequency). In fact, when the magnetic field is vertical, the local vertical motion associated with the radial pulsations (or a low-degree mode) will not be affected and we recover the acoustic cut-off frequency. As the magnetic field becomes more inclined in relation to the local vertical, the necessary frequency for propagation of acoustic waves in the atmosphere decreases.

Concerning the magnetic component, we find that $k^2_{s}$ is positive for oscillations with frequencies larger than $\sigma_{m}$, where

$$\sigma_m = \frac{c}{H} \cos \alpha \left[ \frac{1}{2} - \frac{1}{4(1 + \frac{B^2}{\gamma})} \right]^{1/2}.$$  

The frequency $\sigma_{m}$ increases with height in the atmosphere, taking a maximum value (as $\eta \to -\infty$) of

$$\hat{\sigma}_m = \frac{\sqrt{c}}{2H} \cos \alpha.$$  

In Fig. 4, we show the frequencies $\sigma_{c}$ and $\hat{\sigma}_m$ for the model used in this work.

We emphasize that equations (28) and (29) are approximately valid only in the outer layers, where the magnetic and acoustic components of the wave are decoupled. Hence, they may not be used to directly infer under which conditions the magnetoacoustic wave is reflected (or partially reflected) in the layers below. For that reason, we avoided using the term ‘critical frequencies’ when referring to $\sigma_{c}$ and $\sigma_{m}$. In Section 6, we will look at the problem of wave reflection and pulsation energy losses more closely, using the numerical solutions obtained for the magnetic boundary layer.

The components of the displacement in the $\beta^-$ region can be obtained from the solutions to equations (28) and (29). The general solution to equation (31) is given by

$$\Xi_{1s} = A e^{\hat{\sigma}_{1s} \eta} + B e^{-\hat{\sigma}_{1s} \eta},$$

where $\hat{A}$ and $\hat{B}$ are depth-independent amplitudes. Since no energy can be sent in from outside the star, this must represent a wave propagating outwardly through the isothermal atmosphere. Thus, $\hat{B}$ must be zero. Using equation (30) we thus find the solution to equation (28) to be

$$\varepsilon_{1s} = \frac{\hat{A}}{\rho^{1/2}} e^{\hat{\sigma}_{1s} \eta}.$$  

To find an approximate solution to equation (29), we first note that when terms of the order of $\beta^2$ are neglected, $k^2_{s}$ becomes

$$k^2_{s} \approx \frac{\rho}{|\mathbf{b}_0|^2} (\sigma^2 - \hat{\sigma}_{m}^2).$$

Using $\rho$ as the independent variable, equation (34) thus becomes

$$\rho^2 \frac{d^2 \Xi_{s}}{d\rho^2} + \rho \frac{d\Xi_{s}}{d\rho} + \frac{\rho H^2}{|\mathbf{b}_0|^2} (\sigma^2 - \hat{\sigma}_{m}^2) \Xi_{s} = 0.$$  

The solution to equation (42) can be written in terms of Bessel functions $J_0$ and $Y_0$ (e.g. Abramowitz & Stegun 1972, pg 362), namely,

$$\Xi_{s} = \hat{C} J_0(2\sqrt{\lambda} \rho) + \hat{D} Y_0(2\sqrt{\lambda} \rho),$$

where $\hat{C}$ and $\hat{D}$ are depth-independent amplitudes and $\lambda = H^2|\mathbf{b}_0|^{-2}(\sigma^2 - \hat{\sigma}_{m}^2)$. Since $Y_0$ diverges as $\rho \to 0$, $\hat{D}$ must be zero.
Thus, using equation (33), we find that within the approximations described, the solution to equation (29) has the form
\[
\varepsilon_{\perp} = \frac{\hat{c}}{(p + \hat{B}^2)^{1/2}} J_0(2\sqrt{\hat{\chi} \rho}).
\] (44)

5 SOLUTIONS IN THE MAGNETIC BOUNDARY LAYER

Throughout most of the magnetic boundary layer, the displacement is associated with a magnetoacoustic wave which cannot be described, not even approximately, by decoupled magnetic and acoustic components. Thus, to determine the form of the displacement in this case it is necessary to solve the system of equations (13) and (14) numerically applying, simultaneously, appropriate boundary conditions. In the absence of a magnetic field, the acoustic cut-off frequency is equal to the polar value of \(\nu\) (e.g. Cunha 2006). To derive the two additional boundary conditions will be applied in the boundary layer. In the case of a cyclic frequency well above \(\nu\) (14) numerically applying, simultaneously, appropriate boundary conditions. Thus, using equation (33), we find that within the approximations described, the solution to equation (29) has the form
\[
\varepsilon_{\perp} = \frac{\hat{c}}{(p + \hat{B}^2)^{1/2}} J_0(2\sqrt{\hat{\chi} \rho}).
\] (44)

5.1 Boundary conditions

Since the system of equations to be solved is linear, in order to solve it we need to apply three boundary conditions. Two of these conditions will be applied in the \(\beta^-\) region while the third will be applied in the \(\beta^+\) region.

At the outermost point of the atmosphere, we match the perturbed magnetic field into a vacuum field. This condition implies that \(\varepsilon_{\perp} = 0\) (e.g. Cunha 2006). To derive the two additional boundary conditions, we use the fact that two approximate analytical solutions can be obtained, one in each of the decoupling regions. In the absence of a magnetic field, the acoustic cut-off frequency is equal to the polar value of \(\sigma_\perp\). Since our goal is to investigate on the reflection of oscillations of which frequency is above this acoustic cut-off frequency, in what follows we will consider the case of a cyclic frequency well above \(c/(2\pi 2\mu_0)\).

The complex coefficients \(c_1\) and \(c_2\) are determined by applying the two remaining boundary conditions, that is, by matching simultaneously the relevant components of the numerical solution to the analytical solution (22) with \(E = 0\) in the deeper layers and to the analytical solution (40) in the atmosphere. When performing these matchings, we introduced an integer index \(q\) which reflects the positions at which the matchings are carried out in a grid of depths. The value of this index decreases from a thousand to zero as the depths at which the matchings are carried out are moved from the region where \(\beta \sim 1\) to the regions where the magnetic and acoustic wave components are decoupled (see Appendix A for details). As mentioned in the previous section, we can check the decoupling of the magnetoacoustic wave in the \(\beta^-\) and \(\beta^+\) regions numerically. If such decoupling occurs, the coefficients \(c_1\) and \(c_2\) must tend to constant values as the index \(q\) tends to zero.

Fig. 5 shows the behaviour of \(c_1\) and \(c_2\) when the matchings are carried out at different depths. The results shown are for an oscillation with cyclic frequency \(v = 3.08\) mHz and a dipolar magnetic field of polar strength \(B_p = 3000\) G at a latitude such that its inclination with respect to the local horizontal axis is \(\alpha = 45^\circ\). When the matchings are carried out in the region where \(\beta \sim 1\), which in Fig. 5 corresponds approximately to \(q\) ranging between 600 and

\[c_1 = c_1(q)\]
\[c_2 = c_2(q)\]

Figure 5. The figure shows the behaviour of the complex coefficients of the linear combination expressed by equation (45). A high value of the index \(q\) means that the matchings are carried out in a region where \(\beta \sim 1\), while low values of that index mean that the matchings are carried out in the regions \(\beta^-\) and \(\beta^+\). \(c_1\) and \(c_2\) tend to constant values as the index \(q\) tends to zero, confirming the decoupling of the magnetoacoustic components both in the outer atmosphere and in the interior. Results are for a dipolar magnetic field with polar strength \(B_p = 3000\) G at a latitude such that its inclination with respect to the local horizontal axis is \(\alpha = 45^\circ\), and a cyclic frequency \(v = 3.08\) mHz.

1000, the coefficients \( c_1 \) and \( c_2 \) vary significantly with the index \( q \). This is because the analytical solutions used in the matchings are not good representations of the true solutions in the regions considered. However, as we move the matchings simultaneously to the regions \( \beta^- \) and \( \beta^+ \) (close to \( q = 0 \)) \( c_1 \) and \( c_2 \) tend to constant values, confirming the expected decoupling of the magnetoacoustic wave in both regions.

5.2 Numerical solutions

Having calculated the constants for the linear combination, we can determine the total solution for the displacement. For illustration, we show in Fig. 6 the real part of dimensionless displacement for a dipolar magnetic field of polar strength \( B_p = 3000 \) G, at a latitude such that \( \alpha = 45^\circ \), and a cyclic oscillation frequency \( \nu = 3.08 \) mHz. The figure shows the components of the dimensionless displacement perpendicular and parallel to the direction of the magnetic field (upper panels) and the components of the dimensionless displacement in the vertical and horizontal directions (lower panels).

The decoupling of the magnetoacoustic wave into acoustic and magnetic components both in the \( \beta^- \) and \( \beta^+ \) regions can be seen in the plots. To illustrate that, zooms of Fig. 6 in the interior and in the atmosphere are shown in Figs 7 and 8, respectively. Fig. 7 shows the vertical (upper panel) and horizontal (lower panel) solutions in the interior. The decoupled acoustic and magnetic waves are readily seen. The horizontal component is a rapidly varying magnetic oscillation, while the vertical component is an acoustic oscillation with a significantly larger wavelength. Moreover, an inspection of the real and imaginary parts of the horizontal displacement confirm the running nature of the magnetic wave in the interior, characterized by real and imaginary parts of similar amplitude and out of phase by \( \pi/2 \). The nearly standing nature of the acoustic wave in the interior is also clear from the imaginary part of the vertical displacement, which amplitude tends to zero with increasing depth (alternatively, a standing wave could have real and imaginary parts with significant amplitude, but in phase). Fig. 8 shows the component of the dimensionless displacement perpendicular to the magnetic field direction (upper panel) and the component of the same vector along the magnetic field direction (lower panel), in the atmosphere. As before, the acoustic and magnetic components are easily identified. The component along the direction perpendicular to the magnetic field tends to a constant, as expected for the magnetic wave from the analysis carried out in Section 4.2, while the component of the dimensionless displacement along the direction of the magnetic field is an acoustic oscillation with a wavelength comparable to the thickness of the

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Figure 6. The real parts of the dimensionless displacement perpendicular to the magnetic field (top left-hand panel), along the magnetic field (top right-hand panel), in the vertical direction (bottom left-hand panel), and in the horizontal direction (bottom right-hand panel). Results are for a dipolar magnetic field with polar strength \( B_p = 3000 \) G at a latitude such that its inclination with respect to the local horizontal axis is \( \alpha = 45^\circ \), and a cyclic frequency \( \nu = 3.08 \) mHz.

Figure 7. The dimensionless displacement in the interior. The top panel shows the vertical component while the lower panel shows the horizontal component. The full and dash–dotted lines show, respectively, the real and imaginary parts of the displacement. Results are for the same magnetic field and the same frequency as in Fig. 6.
atmospheric layer shown. Once again, an inspection of the real and imaginary parts of the displacement confirms the running nature of the acoustic wave in the atmosphere, and the standing nature of the magnetic wave in the same region. We note that although the standing nature of the magnetic wave in the atmosphere is connected to the condition of matching of the perturbed magnetic field on to a vacuum field, the form of this component of the displacement is rather insensitive to the place at which that boundary condition is applied, so far as the latter is applied sufficiently high in the atmosphere. Also, as will be discussed in Section 6, this nature of the magnetic wave is not associated to a full wave reflection at a particular place in the atmosphere, but rather to a progressive decrease of its energy content with height.

Figs 6–8 illustrate also the fact that in the atmosphere the decoupling takes place in the parallel and perpendicular directions, while in the interior it takes place in the horizontal and vertical directions, instead.

6 ENERGY FLUXES

6.1 Flux carried by running waves

The acoustic waves that propagate outwardly in the atmospheres of roAp stars and the magnetic waves that propagate inwardly in their interiors, both take energy away from the global oscillations. This is true both for oscillations with frequencies below and above the acoustic cut-off frequency. For oscillations with frequencies below the acoustic cut-off frequency, energy losses through running waves in the atmosphere are restricted to regions where the inclination of the magnetic field is such that \( \sigma > \sigma_c \). On the other hand, oscillations with frequencies above the acoustic cut-off are expected to lose energy through running waves in the atmosphere regardless of the inclination of the magnetic field. For this reason, the observation in some roAp stars of oscillations with frequencies above the acoustic cut-off frequency has been a matter of long debate. At first sight, one would be led to think that perturbations of such high frequency would simply propagate away in the atmosphere and be dissipated before they could grow to observable amplitudes. However, the simple fact that a global oscillation of such high frequency is observed indicates that only part of the mode energy is removed by the running acoustic and magnetic waves. Moreover, it also indicates that the energy input to such a mode in each cycle, through the opacity mechanism acting in the region where hydrogen is ionized (Balmforth et al. 2001; Cunha 2002; Saio 2005), is sufficient to compensate all sources of mode energy losses, including the energy removed by the running waves.

With the question of mode reflection in mind, in this section we estimate the fraction of energy flux that is carried away by running waves and its dependence on the inclination of the magnetic field. As before, the calculations are carried out in a local plane-parallel approach, assuming a dipolar magnetic field and using the numerical solutions for the displacement obtained in Section 5 to compute the relevant energy fluxes at each co-latitude.

Let us consider a wave that propagates outwardly from the interior of the star carrying a given energy flux (see Fig. 9). At some point the wave is reflected, or partially reflected, and propagates inwardly again, back to the starting point. As a result of the magnetoacoustic coupling near the surface layers, this wave transfers part of its energy to a inwardly propagating magnetic wave which eventually dissipates in the \( \beta^+ \) region. Moreover, if the oscillation frequency is greater than the frequency defined by equation (36), part of the wave energy is transferred to an outwardly propagating acoustic wave.

Since the layer where the acoustic and the magnetic components are coupled is thin, and the imaginary part of the frequency is expected to be very small (e.g. Cunha 1999), the amplitude of the oscillation changes only very slightly during the time that takes the wave to propagate through the coupling layer. Therefore, using energy conservation expressed in terms of the dimensionless variables we can write, approximately,

\[
F_{a+}(\eta^*_1, \tau_0) - F_{a+}(\eta^*_1, \tau_0) \approx F_{a+}(\eta^*_2, \tau_0) + F_{m+}(\eta^*_1, \tau_0) \tag{46}
\]

where \( \tau_0 = \tau_{00} \) and \( \tau_0 \) is the time at which the outwardly propagating wave crosses the height \( \eta = \eta^*_1 \). Moreover, \( F_{a+} \), \( F_{m+} \), and \( F_{a+} \) are the dimensionless energy fluxes averaged over one oscillation period carried in the decoupled regions by the outwardly
propagating acoustic component below the coupling layer, the inwardly propagating acoustic component in the same region, the outwardly propagating acoustic wave above the coupling layer and the inwardly propagating magnetic wave in the interior, respectively. Here, the dimensionless energy fluxes are defined as the corresponding energy fluxes divided by the quantity $R^\lambda_0/p_0$.

The average dimensionless energy flux $F_{s+}$ can be calculated using the asymptotic solution for the dimensionless vertical displacement in the $\beta^+$ region given by (e.g. Gough 1993):

$$s_\ast \approx \int \frac{k^{1/2}}{\rho^{1/2}} \cos \left( -\int_\eta^{\eta_0} \kappa d\eta + \delta \right),$$

where $J$ is the depth-independent amplitude, $\delta$ is a complex phase and $k = R\kappa$ with $\kappa$ is the vertical wavenumber of the acoustic oscillation in the interior. Using this asymptotic form of the solution, Cunha & Gough (2000) found for the outwardly propagating wave in the interior

$$F_{s+}(\eta_1^*, \tau_0) \approx \frac{|J|^2 \sigma^2 \epsilon_{k_1} e^{-2\kappa \tau_0}}{8},$$

where the subscripts ‘r’ and ‘i’ stand for the real and imaginary parts of the correspondent quantities.

To determine the fraction of flux that is carried away by the running waves, one needs to find expressions for the energy flux associated with the magnetic wave in the interior and the acoustic wave in the atmosphere. An expression for the former was derived by Cunha & Gough (2000) using the asymptotic solution for the magnetic wave given by equation (22). From the latter, the authors found

$$F_m(\eta_1^*, \tau_0) \approx \frac{|D|^2 \sigma^2 \epsilon_{k_1} e^{-2\kappa \tau_0}}{2\rho},$$

where $D$ is the amplitude of the magnetic wave defined in equation (22).

The flux $\tilde{F}_{s+}$, on the other hand, was not considered by the same authors, who adopted a fully reflective boundary condition at the surface. To derive an expression for the latter, we write the average dimensionless energy flux over one period $P$ carried by the outwardly propagating acoustic wave above the coupling layer in the form

$$\tilde{F}_{s+}(\eta_2^*, \tau_0) \approx \frac{2}{P} \int_0^{\eta_0+P} \int_0^{\tau_0} |v^+|^2 c d\tau,$$

where $v^+$ is the dimensionless velocity of the oscillation, and the dimensionless group velocity of the wave was approximated by $c$. Using equation (40), we write the solution for the outwardly propagating acoustic wave in the $\beta^-$ region in the form

$$\tilde{F}_{s+} = \tilde{F}_m = \frac{\tilde{A}}{p^{1/2}} e^{i \sigma^* \eta} e^\omega t,$$

and, thus,

$$|v^+| \approx |\tilde{A}| \sigma^* e^{-\sigma^* \eta} \sin(k_1 \eta + \sigma^* \tau),$$

where we have neglected the small imaginary part of the frequency in the amplitude, when differentiating the displacement. The corresponding dimensionless energy flux averaged over one period then becomes

$$\tilde{F}_{s+}(\eta_2^*, \tau_0) \approx \frac{|\tilde{A}|^2 \sigma^2 \rho c}{2p} e^{-2\kappa \tau_0}.$$

Fig. 10 shows the fraction of energy flux carried by the running acoustic wave in the atmosphere and by the running magnetic wave in the interior. The amplitudes $|J|, |D|$ and $|\tilde{A}|$ were obtained from the numerical solutions. The results shown are for a dipolar magnetic field with polar strength $B_p = 3000$ G and a cyclic oscillation frequency $\nu = 3.08$ mHz.
maximum of energy flux carried by the running magnetic wave in the interior depends on the frequency of the oscillation.

One interesting aspect to note in Fig. 10 is that the sum of the energy fluxes carried by the running waves is rather large (close to 1) for \( \cos \theta > 0.8 \). In practice, this indicates that for frequencies above the acoustic cut-off, a significant fraction of the energy input in each cycle through the opacity mechanism is lost predominantly in the regions where the magnetic field is close to vertical.

### 6.2 Reflection of the magnetic wave in the atmosphere

As mentioned above, for frequencies above the acoustic cut-off a significant fraction of energy flux is lost near the magnetic poles through running acoustic waves in the atmosphere and running magnetic waves in the interior. Nevertheless, the fact that global waves with frequencies above the acoustic cut-off frequency are observed in some roAp stars indicates that at least part of the mode energy is conserved in the oscillation over each cycle. In this context, it is important to note that the amplitude of the magnetic component of the wave in the atmosphere at a given co-latitude is approximately constant (e.g. Fig. 8). Consequently, the total energy of this wave component must decrease with height, which means that the wave is progressively reflected. To check this idea, we divide the magnetic wave in the atmosphere for different heights in the atmosphere. Besides the rapid decline in the fraction of energy flux carried by the running waves is rather large (close to 0.8) above the acoustic cut-off, a significant fraction of the energy input at frequencies above the acoustic cut-off is expected to be most important. For frequencies above the acoustic cut-off, we have found that the maximum energy losses due to running waves are expected to be most important. For frequencies above the acoustic cut-off, we have found that the maximum energy losses due to running acoustic waves in the atmosphere and running magnetic waves in the interior take place in regions where the magnetic field is not significantly inclined, which for the dipolar magnetic field considered here happens around the magnetic poles.

In this work, we have used a toy model composed of an isothermal atmosphere matched on to a polytropic interior to investigate on the effect that mode conversion may have on the reflection of oscillations near the surface of roAp stars and also to determine the conditions under which energy losses due to running waves are expected to be most important. For frequencies above the acoustic cut-off, we have found that the maximum energy losses due to running acoustic waves in the atmosphere and running magnetic waves in the interior take place in regions where the magnetic field is not significantly inclined, which for the dipolar magnetic field considered here happens around the magnetic poles.

The primary motivation of this study was the understanding of the presence of oscillations with frequencies above the acoustic cut-off in some roAp stars. Hence, the study did not consider oscillations with frequencies below that limit. Nevertheless, from the present results and the results found by previous studies such as those of Cunha & Gough (2000), Saio (2005) and Cunha (2006), which considered full reflection of the waves in the atmosphere, one can easily extrapolate the results that would be obtained when considering frequencies below the acoustic cut-off. In fact, for oscillations with frequency below the acoustic cut-off, the acoustic waves in the atmosphere will propagate only if the magnetic field is sufficiently inclined in relation to the local vertical. Thus, the results for oscillation with such frequencies are expected to resemble those found here, except for the fact that the fraction of energy carried away by the running acoustic wave in the atmosphere will decrease sharply to zero when the magnetic field inclination in relation to the local vertical becomes smaller than the value needed to ensure the
propagation of the acoustic waves in the atmosphere. Thus, in such case, the maximal energy loss due to the presence of running acoustic waves should take place, not at the magnetic pole, but in an annulus around the latter, where the inner radius of that annulus is defined by the minimum inclination of the magnetic field needed for the propagation of the acoustic waves in the atmosphere.

Despite the energy losses associated with the running waves discussed above, our results show that part of the energy is kept in the oscillation every cycle. In fact, the energy associated with the magnetic wave in the atmosphere decreases significantly with height, which means that the wave is progressively reflected. That energy, which is kept in the oscillation, is concentrated in regions where the magnetic field is significantly inclined in relation to the local vertical, that is, around the magnetic equator. Naturally, for oscillations with frequencies below the acoustic cut-off, additional energy would be kept in the oscillation around the magnetic poles, where the acoustic component of the atmospheric wave would be evanescent.

This study considered adiabatic pulsations only, and it is beyond our scope to analyse the impact of the energy losses discussed here on the excitation of the oscillations in roAp stars. Balmforth et al. (2001) carried out linear non-adiabatic calculations of pulsations in roAp stars imposing a transmissive boundary condition at the top when the oscillation frequencies were above the acoustic cut-off. However, in their calculations the authors did not consider the direct effect of the magnetic field on pulsations, which means they ignored both the energy associated with the magnetic wave in the atmosphere, which is kept in the oscillation, and the energy losses associated with the magnetic field in the interior. On a more recent study, Saio (2005) considered linear non-adiabatic pulsations in models of roAp stars taking into account the direct effect of the magnetic field on the oscillations. However, the author imposed a fully reflective outer boundary condition, which means he ignored the energy losses associated with running acoustic waves in the atmosphere. Clearly what is needed is a study of non-adiabatic pulsations with frequencies below the acoustic cut-off, additional energy would be kept in the oscillation around the magnetic poles, where the acoustic component of the atmospheric wave would be evanescent.

APPENDIX A: MATCHING PROCEDURE

Since our calculations are linear, the amplitude of the solution to equations (13) and (14) cannot be determined. Thus, we carry out the matchings necessary to apply the outer and inner boundary conditions by considering the relations

$$\frac{e_1'}{e_1} = \frac{ik_1 - p'}{2p}$$

with $k_1^2 > 0$, \hspace{1cm} (A1)

and

$$\frac{e_1'}{e_1} = -i \left( \frac{\rho \sigma^2}{b_1^2} \right)^{1/2},$$

(A2)

derived, respectively, from the solutions expressed by equations (40) and (22), the first of which is valid in the $\beta^-$ region and the second of which is valid in the $\beta^+$ region.

To compute the left-hand side of the expressions given above, we consider the three independent solutions computed numerically, and combine them according to the relations

$$\frac{e_1'}{e_1} = e_{11}' c_1 + e_{12}' c_2 + e_{13}' c_3$$

(A3)

$$\frac{e_1'}{e_1} = e_{11}' c_1' + e_{12}' c_2' + e_{13}' c_3'$$

(A4)

From these matchings, we determine the values of $c_1$ and $c_2$. In practice, the matchings expressed by equations (A1) and (A2) are carried out for all depths, rather than only in the $\beta^-$ and $\beta^+$ regions. The depths are included in a grid with index $q$, as illustrated in Fig. A1. When $\beta \sim 1$ the index $q$ is close to 1000. As the index $q$ decreases, the matching expressed by equation (A1) is moved towards the $\beta^-$ region and, simultaneously, the matching expressed by equation (A2) is moved to the $\beta^+$ region. Since the analytical solutions used in the matchings are valid only when $q$ is small, only
Figure A1. Grid of depths used in the matchings. A given value of the index $q$ is assigned to a given pair of depths, such that when the pair of depths is in the region $\beta \sim 1$, $q$ is large, and when the pair of depths includes a point in the $\beta^+$ region and a point in the $\beta^-$ region, $q$ is small.

for relatively small values of $q$ one may expect to find $c_1$ and $c_2$ constant. These constants are then used to find the total solution for the displacement across the whole magnetic boundary layer, through the relation expressed in equations (45).

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