Exact Scalar-tensor Cosmologies: Solutions, Asymptotic Behaviour and Dualities

José Pedro Mimoso

Physics Dep. Science Fac. & CAAUL
University of Lisboa

Collaboration with:

Tiago Charters (Inst. Politech. Lisboa & CAAUL)

We investigate a mechanism that generates exact solutions of scalar-tensor cosmologies with a perfect fluid. We work in the so-called Einstein frame, recovering known solutions and obtaining new ones. The method is considerably simpler than previous methods found in the literature, namely the method devised by Barrow and Mimoso. We also discuss the existence of form-invariance dualities that relate pairs of solutions.

Scalar-tensor gravity theories in cosmology

* Kaluza-Klein type theory to underly Dirac's Large Number Hypothesis (P. Jordan . 1940ies)

** Brans-Dicke theory to account for Mach principle of inertia (1961)

*** Non-minimal coupling and Conformally coupled scalar field theory (1968-70)

**** Kaluza-Klein type and String theory unification proposals.

H Nariai (1968) *Prog. Theor. Phys.* **40**, 49

L E Gurevich, A M Finkelstein, V A Ruban (1973) *Astrophys. Space Sci.* **22**, 231

D. Lorenz-Petzold, Astrophys. Space Sci. **98**, 249-254 (1984).

J O'Hanlon, B O J Tupper (1972) *Nuovo Cimento* **7**, 305

J.D. Barrow, *Phys Rev D* **47**, 5329 (1993).

J. D. Barrow and J. P. Mimoso, Phys. Rev. D **50** (1994) 3746.

J. P. Mimoso and D. Wands, Phys. Rev. D **51**, (1995) 477.

J. P. Mimoso and D. Wands, Phys. Rev. D 52, 5612 (1995).

A. Billyard, A.A. Coley and J. Ibañez, Phys. Rev. D 59, 023507 (1999).

A.A. Coley, Gem. Rel. Grav. 31, 1295 (1999).

D. I. Santiago, D. Kalligas and R. V. Wagoner "Scalar-tensor cosmologies and their late time evolution", Report. No. gr-qc/9805044 (1998) (unpublished).

S. S. Yazadjiev, Phys. Rev. D **65**: 084023 (2002).

A. Navarro, A. Serna and J. M. Alimi, Phys. Rev. D **59**:124015 (1999).

J. D. Barrow and D. J. Shaw, ??? [arXiv:0712.2190 [gr-qc]].

J. D. Barrow and P. Parsons, Phys. Rev. D **55**: 1906 (1997).

T. Damour & K. Nordvedt, *Phys. Rev. Lett.* **70**, 2217 (1993); *Phys. Rev. D***48**, 3436 (1993).

J. P. Mimoso & A. Nunes, Phys. Lett. A 248, 325 (1998).

Jos P. Mimoso & Ana Nunes, Astrophysics and Space Science 283, 661-666 (2003).

$$H^2 + H\,\frac{\dot\phi}{\phi} - \frac{\omega(\phi)}{6}\,\frac{\dot\phi^2}{\phi^2} + \frac{k}{a^2} = \frac{8\pi}{3}\,\frac{\rho}{\phi}\,, \tag{5}$$

$$\ddot\phi + \left[3H + \frac{\dot\omega(\phi)}{2\omega(\phi)+3}\right]\dot\phi = \frac{8\pi\rho}{2\omega(\phi)+3}\,(4-3\gamma)\,, \tag{6}$$

$$\dot H + H^2 + \frac{\omega(\phi)}{3}\,\frac{\dot\phi^2}{\phi^2} - H\,\frac{\dot\phi}{\phi}$$

$$= -\frac{8\pi\rho}{3\phi}\,\frac{(3\gamma-2)\,\omega+3}{2\omega(\phi)+3} + \frac{1}{2}\,\frac{\dot\omega}{2\omega(\phi)+3}\,\frac{\dot\phi}{\phi}\,, \tag{7}$$

$$\dot\rho + 3\gamma H\,\rho = 0\,, \tag{8}$$

we can rewrite the above field equations as

$$(X')^2 + 4\,k\,X^2 - (Y'\,X)^2 = 4MXa^{4-3\gamma}\,, \tag{3.8}$$

$$[Y'\,X]' = M(4-3\gamma)\,\sqrt{\frac{3}{2\omega+3}}\;a^{4-3\gamma}\,, \tag{3.9}$$

$$X'' + 4\,k\,X = 3\,M(2-\gamma)\,a^{4-3\gamma}\,, \tag{3.10}$$

where the density $\rho = 3M/8\pi a^{3\gamma}$ for a barotropic fluid with M a constant. The prime denotes differentiation with respect to η. Our variables are akin to those used by Lorenz-Petzold [17] when solving for the Brans-Dicke theory. To that extent the method we explore here is a generalization of his method of obtaining decoupled equations. Note that whenever X is negative this must correspond to a negative value for ϕ. In what follows, unless otherwise explicitly stated, we shall assume that $\omega > -3/2$, to guarantee the positiveness of the function under the square root in Eq. (3.7), although it would be straightforward to redefine $Y(\phi)$ for the case of $\omega < -3/2$.

This system considerably simplifies for the two partic-

Eqs. (5)–(7) and a late period during which the solutions asymptote towards the matter-dominated power-law behavior. In what follows no restrictive assumption will be made regarding the boundary conditions satisfied by $\phi(t)$ at $t = 0$ and so our analysis will be kept fully general.

The system of equations (5)–(8) allows considerable simplification in the cases where the trace of the energy-momentum tensor vanishes and the $\omega(\phi)$ theories are conformally related to general relativity [11]. In these cases, the wave equation (6) is sourceless and the general relativity solutions always arise as a particular ($\phi =$ const) case of the general solutions. By exploiting this situation, exact vacuum and radiation cosmological solutions for scalar-tensor gravity theories for all values of k have been found by one of us [14].

To address the remaining nonvacuum fluid cases ($\gamma \neq 4/3$) we resort to a generalization of the method of integration used by Gurevich *et al.* [20] in their study of Brans-Dicke (BD) theory to the more complicated cases where ω depends on ϕ. Our procedure will be applied to the $k = 0$ models.

We introduce a new time variable η via

$$dt = a^{3(\gamma-1)} \sqrt{\frac{2\omega+3}{3}}\, d\eta \tag{9}$$

(we assume $2\omega + 3 > 0$) and two new dynamical variables

$$x \equiv \left[\phi a^{3(1-\gamma)} \frac{d}{d\eta} a^3\right] , \tag{10}$$

$$y \equiv \left[a^{3(2-\gamma)} \frac{d}{d\eta} \phi\right] . \tag{11}$$

The $k = 0$ field equations (6) and (7) reduce to

$$y' = M\,(4 - 3\gamma) \tag{12}$$

and

$$x' = 3M\,[(2-\gamma)\omega + 1] + \frac{3}{2}\left(\frac{2}{3}x + y\right) \frac{\omega'}{2\omega + 3} , \tag{13}$$

where the prime denotes differentiation with respect to η and M is defined by $8\pi\rho = 3Ma^{-3\gamma}$. In addition to these equations, the Friedmann equation (5) yields the constraint

$$\left(\frac{2}{3}x + y\right)^2 = \left(\frac{2\omega+3}{3}\right) \left[y^2 + 4M\,\phi\, a^{3(2-\gamma)}\right] . \tag{14}$$

It is straightforward to integrate Eqs. (12) and (13). The solutions are

$$y = M(4 - 3\gamma)\,(\eta - \eta_1) \quad , \tag{15}$$

$$x = \frac{3}{2}\left[-y + \sqrt{2\omega+3}\left(C + M(2-\gamma) \times \int_{\eta_1}^{\eta} \sqrt{2\omega+3}\, d\bar{\eta}\right)\right] , \tag{16}$$

where η_1 and C are integration constants.

Now we differentiate y using the definition (11) and

$$3\frac{a'}{a}\frac{\phi'}{\phi} = \frac{(\phi')^2}{\phi^2}\, 3\frac{a'}{a}\frac{\phi}{\phi'} = \frac{(\phi')^2}{\phi^2}\frac{x}{y} \tag{17}$$

to get

$$\left(\frac{\phi'}{\phi}\right)' + \left[\frac{3\gamma-4}{2} + \frac{f'(\eta)}{\eta - \eta_1}\right] \left(\frac{\phi'}{\phi}\right)^2 = \frac{1}{\eta-\eta_1}\left(\frac{\phi'}{\phi}\right) , \tag{18}$$

where we have defined the function

$$f(\eta) \equiv \int_{\eta_1}^{\eta} \frac{3(2-\gamma)}{2M(4-3\gamma)} \sqrt{2\omega(\phi)+3} \times \left[C + M(2-\gamma) \int_{\eta_1}^{\bar{\eta}} \sqrt{2\omega(\phi)+3}\, d\bar{\bar{\eta}}\right] d\bar{\eta} \;. \tag{19}$$

By introducing another function $g(\eta)$ we can absorb $f(\eta)$ into

$$g(\eta) \equiv f(\eta) + \frac{3\gamma-4}{4}(\eta - \eta_1)^2 + D, \tag{20}$$

where D is the integration constant arising from solving the Bernoulli equation (18). The solutions to Eq. (18) can be cast into the particularly simple form

$$\ln\left(\frac{\phi}{\phi_0}\right) = \int_{\eta_1}^{\eta} \frac{\eta - \eta_1}{g(\eta)} d\eta \tag{21}$$

and

$$a^3 = a_0^3 \left(\frac{g}{\phi}\right)^{\frac{1}{2-\gamma}} , \tag{22}$$

which follows from Eq. (17). In terms of $f(\eta)$ the behavior of the coupling $\omega(\phi)$, which defines the theory, is given by

$$2\omega(\phi(\eta)) + 3 = \frac{4-3\gamma}{3(2-\gamma)^2} \frac{(f')^2}{\left[f + \frac{4-3\gamma}{3(2-\gamma)^2} f_0\right]}, \tag{23}$$

where f_0 is another arbitrary constant. The $\omega(\phi)$ dependence is obtained by solving Eq. (21) with respect to ϕ, whenever this is possible. In practice a theory can be chosen by specifying the generating function $g(\eta)$ from which $\phi(\eta)$ follows from Eq. (21) and hence $a(\eta)$ from Eq. (22), $f(\eta)$ from Eq. (20), $\omega(\phi)$ from Eq. (23), and $\eta(t)$ from Eq. (9) if all the integrals can be performed.

It is worth noticing that the constant η_1, which was introduced in Eq. (15), can be set equal to zero without loss of generality. This merely amounts to a translation of the origin of the time variable. Henceforth, we use this freedom and take $\eta_1 = 0 = t$.

Flat Friedmann Universe with one scalar field

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3}(dx^i)^2$$

General Relativity

$$3\frac{\dot{a}^2}{a^2} = \frac{\dot{\phi}^2}{2} + V(\phi)$$

$$3\frac{\ddot{a}}{a} = -(\dot{\phi}^2 - V)$$

$$\dot{\rho}_\phi = -3H\dot{\phi}^2$$

Scalar-tensor theories

$$\frac{3\dot{a}^2}{a^2} = m(\phi)\rho + \frac{\dot{\phi}^2}{2} + V(\phi)$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V(\phi)}{\partial \phi} = -\rho\frac{\partial m(\phi)}{\partial \phi}$$

$$\dot{\rho} = -3H(\rho + p) + \rho\frac{\partial m}{\partial \phi}\dot{\phi}$$

$$m(\phi) = m_0 \exp\left(-\sqrt{2}(3\gamma - 4)\,\alpha\right)$$

Definining new variables

$$\frac{3\dot{a}^2}{a^2} = \frac{\dot{\phi}^2}{2} + m(\phi)a^{-3\gamma}$$

$$\dot{H} = -\frac{\dot{\phi}^2}{2} - \frac{\gamma}{2} m(\phi) a^{-3\gamma}$$

$$x = \frac{\dot{\phi}}{H}$$

$$y = \frac{\sqrt{m(\phi)a^{-3\gamma}}}{\sqrt{3}H}$$

$$\Omega = 1 = \left(\frac{x^2}{6} + y^2 \right)$$

Only one variable is independent

x(Φ)

$$y^2 = 1 - \frac{x^2}{6}$$

Autonomous dynamical systems for STT

$$x' = -\left(3 - \frac{x^2}{2}\right)\left(\left(1 - \frac{\gamma}{2}\right) x + \frac{\partial_\phi m}{m}\right)$$

$$\phi' = x$$

[Nunes & Mimoso PLB 2000]

This generalises to STT the use of Φ

In GR [Muslimov 1989, Lidsey 1991,Salopek and Bond, Charters Mimoso 2009]

Everything reduces to one ODE
With the coefficients slightly modified

$$x \frac{dx}{d\phi} = -\left(3 - \varepsilon \frac{x^2}{2}\right)\left(x + \varepsilon \frac{\partial_\phi m}{m}\right)$$

Choosing x(Φ) we obtain "everything"

$$m(\phi) = m_0\left(3 - \frac{x^2(\phi)}{2}\right)\exp\left(-\int x(\phi)\,d\phi\right)$$

$$H(\phi)\,a^{\frac{3\gamma}{2}} = \pm\sqrt{m_0}\,\exp\left(-\frac{1}{2}\int x(\phi)\,d\phi\right)$$

$$a = a_0\exp\left(\int\frac{d\phi}{x(\phi)}\right)$$

$$t = \int\frac{d\phi}{x(\phi)H(\phi)}$$

$$\dot{\phi}^2 = x^2(\phi)\exp\left(-\int x(\phi)\,d\phi\right)$$

Catalogue of exact solutions

Some examples

$$x(\phi) = \lambda = cte$$

Brans-Dicke theory

$$x(\phi) = \phi$$

$$2\omega + 3 \cong \left(\frac{2}{\lambda^2} \left(\frac{4 - 3\gamma}{2 - \gamma} \right)^2 \right) \ln^{-1} \left(\frac{\Phi}{\Phi_0} \right)^{\frac{2}{\lambda}\left(\frac{4-3\gamma}{2-\gamma}\right)}$$

$$x(\phi) = \frac{\lambda}{\phi}$$

$$2\omega + 3 \cong \left(\frac{2}{\lambda^2} \left(\frac{4 - 3\gamma}{2 - \gamma} \right)^2 \right) \times \left(\frac{\Phi}{\Phi_0} \right)^{\frac{2}{\lambda}\left(\frac{2-\gamma}{4-3\gamma}\right)}$$

$$x(\phi) = \beta \tan \lambda\phi$$

Conformally coupled s.f.

$$2\omega + 3 \cong \frac{3}{1 - \left(\frac{\Phi}{\Phi_0}\right)}$$

Barker's theory

$$\omega \cong \frac{4 - 3\left(\frac{\Phi}{\Phi_0}\right)}{2\left(1 - \left(\frac{\Phi}{\Phi_0}\right)\right)}$$

Assimptotic behaviour of STT

$$x' = -\left(3 - \frac{x^2}{2}\right)\left(\left(1 - \frac{\gamma}{2}\right) x + \partial_\phi m \big/ m\right)$$

$$\phi' = x$$

Fixed points at finite Φ

$$x = 0 \Rightarrow m'/m = 0$$

i.e. GR, radiation

Fixed points at infinite Φ

$$x = x_* \Rightarrow m'/m = \lambda = const$$

i.e. BD

Duality

$$\phi \to \bar{\phi} = \phi$$

The map

$$x(\phi) = \lambda \to \bar{x}(\bar{\phi}) = \bar{\lambda} = 1/c$$

$$\bar{H}(\phi) = cH(\phi)$$

e.g. c=-1

Now the dualities may relate different scalar-tensor theories!

Conclusions

Exact ST solutions with a barotropic γ- perfect fluid, both known and new, are derived from an adequate choice of a generating function $x(\Phi)$.

In GR, this essentially depends on the equation of state and determines the form of the potential.

The integration procedure which is here produced is a lot simpler than the methods found in the literature so far.

We found a form-invariance duality between two scalar-tensor cosmologies. Hope to extend this to more general cases…

Our procedure also encompasses scalar-tensor gravity theories with a barotropic γ- perfect fluid.

The integration procedure which is here produced is a lot simpler than the methods found in the literature so far.

We found a duality between different scalar-tensor cosmologies. Hope to extend this to more general cases…

THANKS for listening !