5th IBERIAN COSMOLOGY MEETING

30th March 2010, Porto, Portugal

Cosmological perturbations in *f(R)* theories

Álvaro de la Cruz-Dombriz

Theoretical Physics Department

Complutense University of Madrid

in collaboration with

Antonio Dobado & Antonio López Maroto

ON THIS TALK...

I. Overview

- Motivation
- Usual analysis in general relativity + cosmological constant.

II. Perturbed Einstein equations for f(R) theories

- Background equations.
- Viability conditions for *f(R)* theories
- First order perturbed equations. General expressions.
- Sub-Hubble modes and quasi-static approximation.

III . Applications

- Some viable vs. excluded *f(R)* models.
- Conclusions.

I. OVERVIEW

- **Accelerated expansion of the universe,**

  - usually explained through a **cosmological constant Λ.**
  - more generically through a **Dark Energy** contribution.

  } but its nature "remains" ignored

- Quintessence, braneworlds, Scalar-tensor theories, *f(R)* gravities…

- It has been shown that *f(R)* functions can mimic **any expansion history**, in particular that of ΛCDM.

ACD & A. Dobado, PRD 74: 087501, 2006.

- The exclusive use of observations from SNIa, baryon acoustic oscillations or CMB shift factor , are sensitive only to the cosmological expansion history, cannot settle the question of the dark energy nature.

- Research on evolution of perturbations is required to determine and/or distinguish dark energy nature.

CONVENTIONS

Flat FLRW metric: Conformal time + Longitudinal gauge

$$ds^2 = a^2(\eta)[(1+2\Phi)d\eta^2 - (1-2\Psi)(dr^2 + r^2 d\Omega_2^2)]$$

$$\Phi \equiv \Phi(\eta, \vec{x})$$

$$\Psi \equiv \Psi(\eta, \vec{x})$$

Bardeen's potential for cosmological scalar perturbations

- Gravitational action: $S_G = \frac{1}{2\kappa}\int d^4x\sqrt{|g|}[R + f(R)]$

- Density perturbation δ : $\delta\rho = \rho_0\delta$

ρ_0 Unperturbed mean energy density

$\delta\rho = \rho - \rho_0$

GENERAL RELATIVITY + Λ

✓ Varying the **EH + Λ action** with respect to the metric, first order perturbed Einstein equations are found

$\Phi = \Psi$ **The two introduced potentials are the same.**

$$\delta'' + \mathcal{H}\frac{k^4 + 6\tilde{\rho}k^2 - 18\tilde{\rho}^2}{k^4 + \tilde{\rho}(3k^2 - 9\mathcal{H}^2)}\delta' - \tilde{\rho}\frac{k^4 + 9\tilde{\rho}(2\tilde{\rho} - 3\mathcal{H}^2) + k^2(9\tilde{\rho} - 3\mathcal{H}^2)}{k^4 + \tilde{\rho}(3k^2 - 9\mathcal{H}^2)}\delta = 0$$

- **Sub-Hubble modes** $k >> \mathcal{H}$

$$\delta'' + \mathcal{H}\delta' - 4\pi G\rho_0 a^2\delta = 0$$

- Pure Einstein-Hilbert action **$\delta = a$**

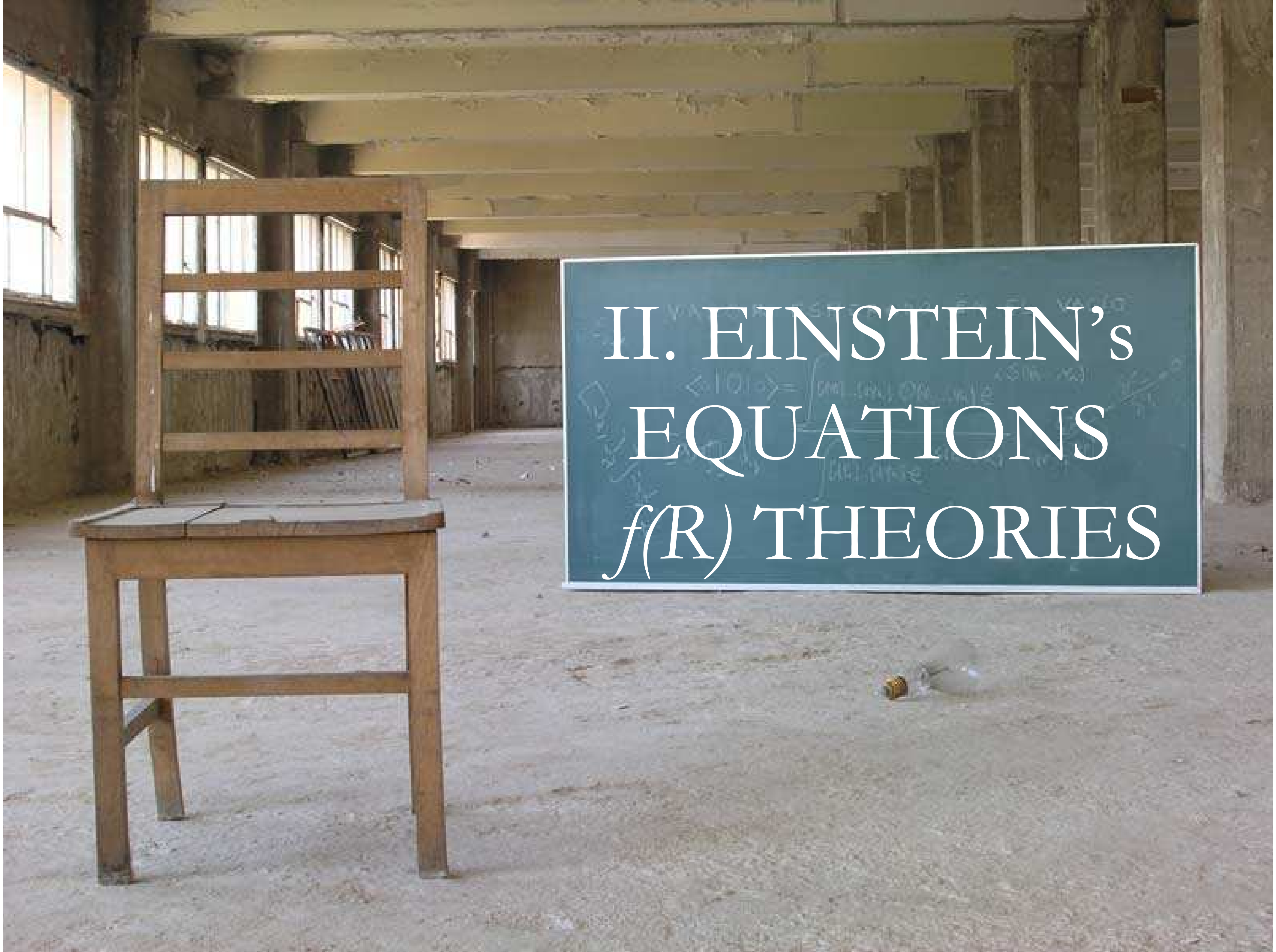
$$\tilde{\rho} \equiv 4\pi G\rho_0 a^2 = -\mathcal{H}' + \mathcal{H}^2$$

- EH + Λ action

$$\frac{\delta(a)}{a} \doteq e^{\int_{a_i}^{a}[\Omega_m(a)^\gamma - 1]\mathrm{d}\ln a}$$

E. Linder.
PRD 72: 043529, (2005).

$\gamma = 6/11$


II. EINSTEIN's
EQUATIONS
f(R) THEORIES

BACKGROUND EQUATIONS

$$(1+f_R)R_{\mu\nu}-\frac{1}{2}(R+f(R))g_{\mu\nu}+\mathcal{D}_{\mu\nu}f_R = -\kappa T_{\mu\nu}$$

$$f_R \equiv \mathrm{d}f(R)/\mathrm{d}R$$

$$\mathcal{D}_{\mu\nu} \equiv D_\mu D_\nu - g_{\mu\nu}\Box$$

✓ FLRW metric:

$$3(1+f_R)\frac{\ddot{a}}{a}-\frac{1}{2}(R+f(R))-3\frac{\dot{a}}{a}\dot{R}f_{RR} = -8\pi G\,\rho_0$$

Third order differential equation in scale factor a !!

✓ It can be proven that any expansion history can be mimicked by a well-chosen *f(R)* model, in particular that of ΛCDM model .

ACD & A. Dobado, PRD 74: 087501, 2006.

VIABLE *f(R)* MODELS

$$f_R \equiv \mathrm{d}f(R)/\mathrm{d}R$$
$$f_{RR} \equiv \mathrm{d}^2 f(R)/\mathrm{d}R^2$$

L. Pogosian & A. Silvestri, PRD 77 023503, (2008).

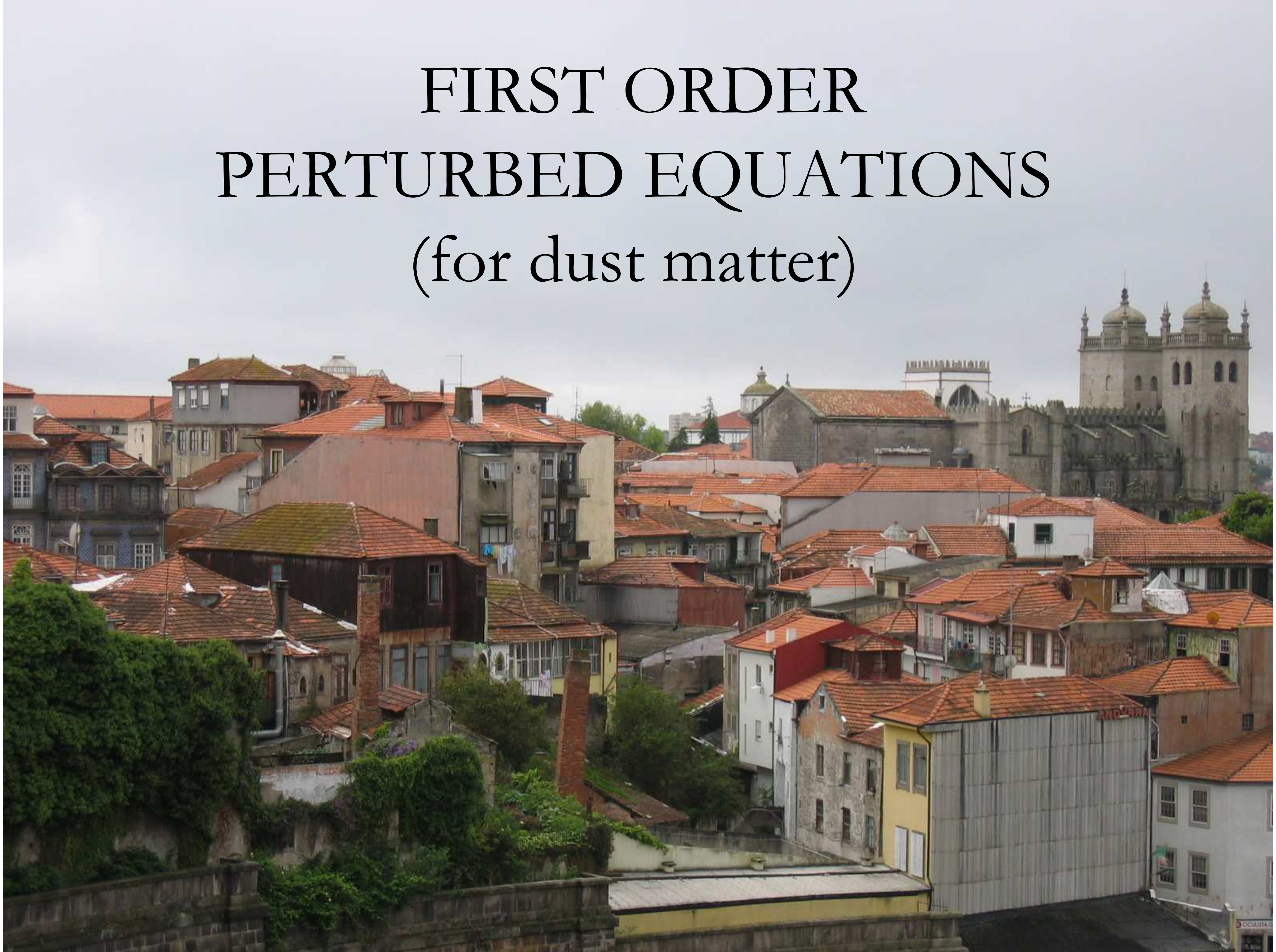
- **I.** $f_{RR} > 0$ for high curvatures **W. Hu & I. Sawicki. PRD 76 064004, (2007).**
  Classically stable high-curvature regime & existence of matter dominated phase.

- **II.** $1 + f_R > 0$ for all R. This condition ensures the effective Newton's constant to be positive at all times and the graviton energy to be positive.

- **III.** $f_R < 0$ ensures that ordinary General Relativity behavior is recovered at early times. Together with the condition $f_{RR} > 0$, it implies that f_R should be in the range $-1 < f_R < 0$.

- **IV.** $|f_{RR}| << 1$ at recent epochs. This is imposed by local gravity tests although it is **still not clear what is the actual limit on this parameter.**
  It also implies that the cosmological evolution at late times resembles that of **ΛCDM**.

FIRST ORDER PERTURBED EQUATIONS (for dust matter)



May one ask if…

- Is still valid the process to reproduce an exact differential equation for δ decoupled from the rest of perturbed quantities?

- Is the differential equation for δ second order?

- Does δ depend on the *f(R)* choosen model?

- For sub-Hubble scales, time derivatives of involved quantities are usually neglected (quasi-static approximation). Is that approximation rigorously valid?

➢ BACKGROUND EINSTEIN EQUATIONS

Combining density & pressure equations

$$2(1+f_R)(-\mathcal{H}'+\mathcal{H}^2)+2\mathcal{H}f_R'-f_R''=8\pi G\rho_0(1+c_s^2)a^2$$

➢ Perturbed conservation equations

$$\delta G^\mu{}_\nu = -8\pi G\delta T^\mu{}_\nu$$

$$\begin{aligned}\delta T^0{}_0 &= \delta\rho = \rho_0\delta \\ \delta T^i{}_j &= -(\delta P)\delta^i{}_j \\ \delta T^0{}_i &= -\delta T^i{}_0 = -(\rho_0+P_0)\partial_i v\end{aligned}$$

$$\tilde{D}_\mu \tilde{T}^\mu_\nu = 0$$

$$\delta P/\delta\rho \equiv c_S^2 \equiv P_0/\rho_0$$

➢ Perturbed Einstein equations

$$\Phi-\Psi = -\frac{f_{RR}}{1+f_R}\delta R$$

Ψ NOT EQUAL TO Φ

$$\delta R = -\frac{2}{a^2}[3\Psi''+6(\mathcal{H}'+\mathcal{H}^2)\Phi+3\mathcal{H}(\Phi'+3\Psi')+k^2(\Phi-2\Psi)]$$

DIFFERENTIAL EQUATION [(δ)]

$$\beta_{4,f}\delta^{iv} + \beta_{3,f}\delta''' + (\alpha_{2,\mathrm{EH}} + \beta_{2,f})\delta'' + (\alpha_{1,\mathrm{EH}} + \beta_{1,f})\delta' + (\alpha_{0,\mathrm{EH}} + \beta_{0,f})\delta = 0$$

- Coefficients are separated in

EH part : From linear part in gravitational action (α**´s**)

f(R) **theory part:** From non-linear part in gravitational action (β**´s**).

involves terms with f'_R and f''_R .

- δ^{iv} and δ''' coefficients DO NOT have **EH part**.

- If *f(R)* **theory part** is removed, usual expressions for **EH** action with/without **Λ** are recovered.

SOME COEFFICIENTS…

✓ For sub-Hubble modes, expansion in parameter ϵ can be performed.

$$\kappa_i \equiv \mathcal{H}'^{(i)}/\mathcal{H}^{i+1} \, (i = 1,2,3)$$

$$\epsilon \equiv \mathcal{H}/k$$

Dimensionless parameters

➢ Coefficients for δ´´term

$$\begin{aligned}
\alpha_{2,\mathrm{EH}}^{(1)} &= 432(1+f_R)^{10}\mathcal{H}^2\epsilon^8(-1+\kappa_1)(-2+\kappa_2)^3 \\
\alpha_{2,\mathrm{EH}}^{(2)} &= -1296(1+f_R)^{10}\mathcal{H}^2\epsilon^{10}(-1+\kappa_1)^2(-2+\kappa_2)^3 \\
\alpha_{2,\mathrm{EH}}^{(3)} &= 3888(1+f_R)^{10}\mathcal{H}^2\epsilon^{12}(-1+\kappa_1)^2(-2+\kappa_2)^3 \\
\beta_{2,f}^{(1)} &\simeq 8f_R^4(1+f_R)^6 f_1^4\mathcal{H}^2 \\
\beta_{2,f}^{(2)} &\simeq -88f_R^3 f_1^3\mathcal{H}^2\epsilon^2(-2+\kappa_2) \\
\beta_{2,f}^{(3)} &\simeq 24f_R^2 f_1^2\mathcal{H}^2\epsilon^4(-2+\kappa_2)(-28+2\kappa_1+13\kappa_2) \\
\beta_{2,f}^{(4)} &\simeq -72f_R f_1\mathcal{H}^2\epsilon^6(-2+\kappa_2)^2(-14+4\kappa_1+5\kappa_2)
\end{aligned}$$

Very rapid *f(R)* part dominance !!

$$\alpha_{i,\mathrm{EH}} = \sum_{j=1}^{3}\alpha_{i,\mathrm{EH}}^{(j)} \quad (i = 0,1,2)$$

SUB-HUBBLE MODES EVOLUTION

✓ After strong simplifications in perturbed Einstein's equations by neglecting temporal derivatives of Φ and Ψ, the equation for δ in sub-Hubble regime becomes

$$\delta'' + \mathcal{H}\delta' - \frac{1 + 4\frac{k^2}{a^2}\frac{f_{RR}}{1+f_R}}{1 + 3\frac{k^2}{a^2}\frac{f_{RR}}{1+f_R}} \frac{\tilde{\rho}\delta}{1+f_R} = 0$$

A. Starobinsky et al., **PRL 85 2236, (2000).**
P. Zhang, **PRD 73 123504, (2006).**
S. Tsujikawa, **PRD 76 023514, (2007).**
A. Starobinsky **JETP L.86:157-163, (2007).**

Quasi static approximation

SUB-HUBBLE MODES EVOLUTION

✓ After strong simplifications in perturbed Einstein's equations by neglecting temporal derivatives of Φ and Ψ, the equation for δ in sub-Hubble regime becomes

$$\delta'' + \mathcal{H}\delta' - \frac{1 + 4\frac{k^2}{a^2}\frac{f_{RR}}{1+f_R}}{1 + 3\frac{k^2}{a^2}\frac{f_{RR}}{1+f_R}}\frac{\tilde{\rho}\delta}{1+f_R} = 0$$

A. Starobinsky et al., **PRL 85 2236, (2000).**
P. Zhang, **PRD 73 123504, (2006).**
S. Tsujikawa, **PRD 76 023514, (2007).**
A. Starobinsky **JETP L.86:157-163, (2007).**

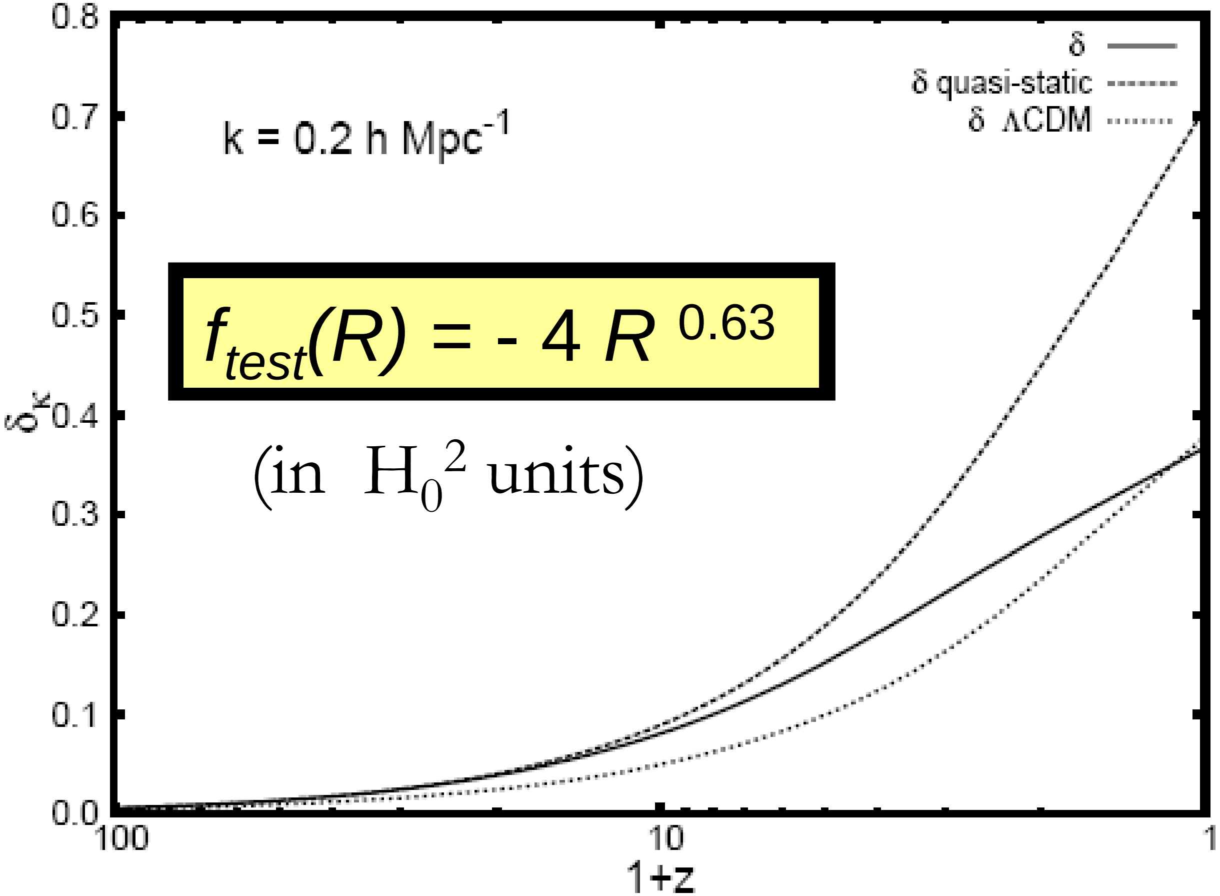
Quasi static approximation

✓ Using our results, if first α and first β are taken, then

$$\delta'' + \mathcal{H}\delta' + \frac{(1+f_R)^5\mathcal{H}^2(-1+\kappa_1)(2\kappa_1-\kappa_2) - \frac{16}{a^8}f_{RR}^4(\kappa_2-2)k^8 8\pi G\rho_0 a^2}{(1+f_R)^5(-1+\kappa_1) + \frac{24}{a^8}f_{RR}^4(1+f_R)(\kappa_2-2)k^8}\delta = 0$$

Note $\boldsymbol{k^8}$ difference in scale dependence so a rapid dominance of *f(R)* contribution ($\boldsymbol{k^8}$ difference instead $\boldsymbol{k^2}$ in Quasi-static approximation).

COMPARISON BETWEEN QUASI-STATIC APPROXIMATION & COMPLETE SUB-HUBBLE LIMIT



$f_{test}(R) = -4\,R^{0.63}$

(in H_0^2 units)

- This function accomplishes all conditions for ***f(R)*** except for **IV**, ie. $|f_R|$ **is not much smaller 1 today.**

-Both evolutions are the same at early times, where **EH** contribution is dominant.

- Both *f(R)* approaches differ from **ΛCDM** predictions.

➢ **At late times, quasi-static approximation FAILS to describe perturbations evolution.**

✓ More careful derivations in sub-Hubble modes, keeping **four β coeffficients & first α coefficients** and considering

$$| f_R | \ll 1$$

Equation **[(δ)]** becomes a second order differential equation

$$\delta'' + \mathcal{H}\delta' - \frac{4}{3}\frac{\left(\frac{6 f_{RR} k^2}{a^2} + \frac{9}{4}\right)^2 - \frac{81}{16} + \frac{9}{2}\frac{2\kappa_1 - \kappa_2}{-2+\kappa_2}}{\left(\frac{6 f_{RR} k^2}{a^2} + \frac{5}{2}\right)^2 - \frac{25}{4} + 6\frac{-1+\kappa_1}{-2+\kappa_2}}(1-\kappa_1)\mathcal{H}^2\delta = 0$$

where $2\kappa_1 - \kappa_2 \approx -2 + \kappa_2 \approx -1 + \kappa_1$

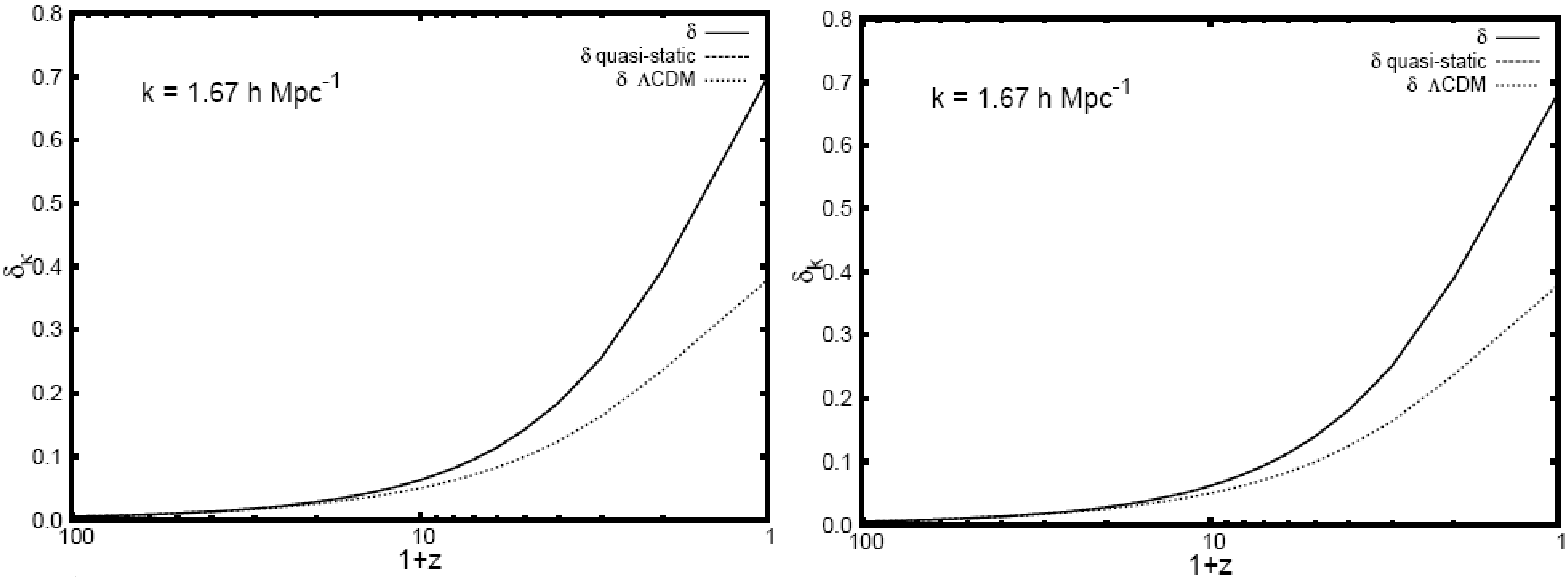
✓ In other words, for general *f(R)* functions the **quasi-static approximation is not justified**.

✓ However for those viable functions describing the present phase of accelerated expansion & satisfying local gravity tests, i.e. $| f_R | \ll 1$, it does give a correct description for the evolution of perturbations.

III. APPLICATIONS: *f(R)* MODELS

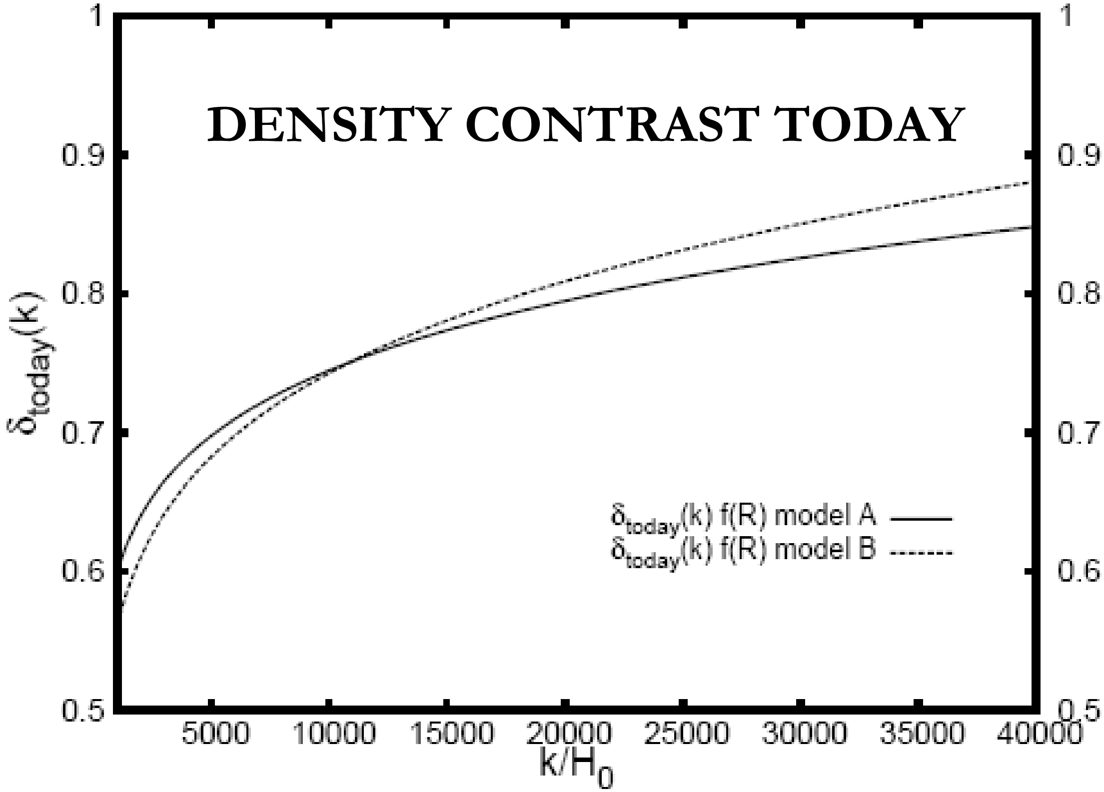
L. Amendola et al., **PRD 75 083504 (2007).**   I. Sawicki and W. Hu, **PRD 75 127502, (2007).**

$f_A(R) = -4.3\, R^{0.01}$ $f_B(R) = (2.5\cdot 10^{-4}\, R^{0.3} - 0.22)^{-1}$



- Both functions follow all viability conditions.
- Quasi-static and exact evolutions are **indistinguishable** but different from ΛCDM at recent times.

VIABLE MODELS: k DEPENDENCE



- Strictly growing behaviour.
- *f(R)* perturbation evolution equation **DOES depend** on scale *k*.
  - k^2 for quasi-static evolution.
  - k^8 for correct deviation.
- In **EH+Λ** sub-Hubble modes evolution does **NOT depend** on scale ***k***.

✓ Matter power spectrum

$$P_k^{f(R)} = T(k)\, P_k^{\Lambda CDM}$$

- Transfer function $T(k)$ is ***k*** dependent.
- $P_k^{f(R)}$ is different from $P_k^{\Lambda CDM}$.

Viable model?

Miranda et al.

PRL 102:221101, (2009).

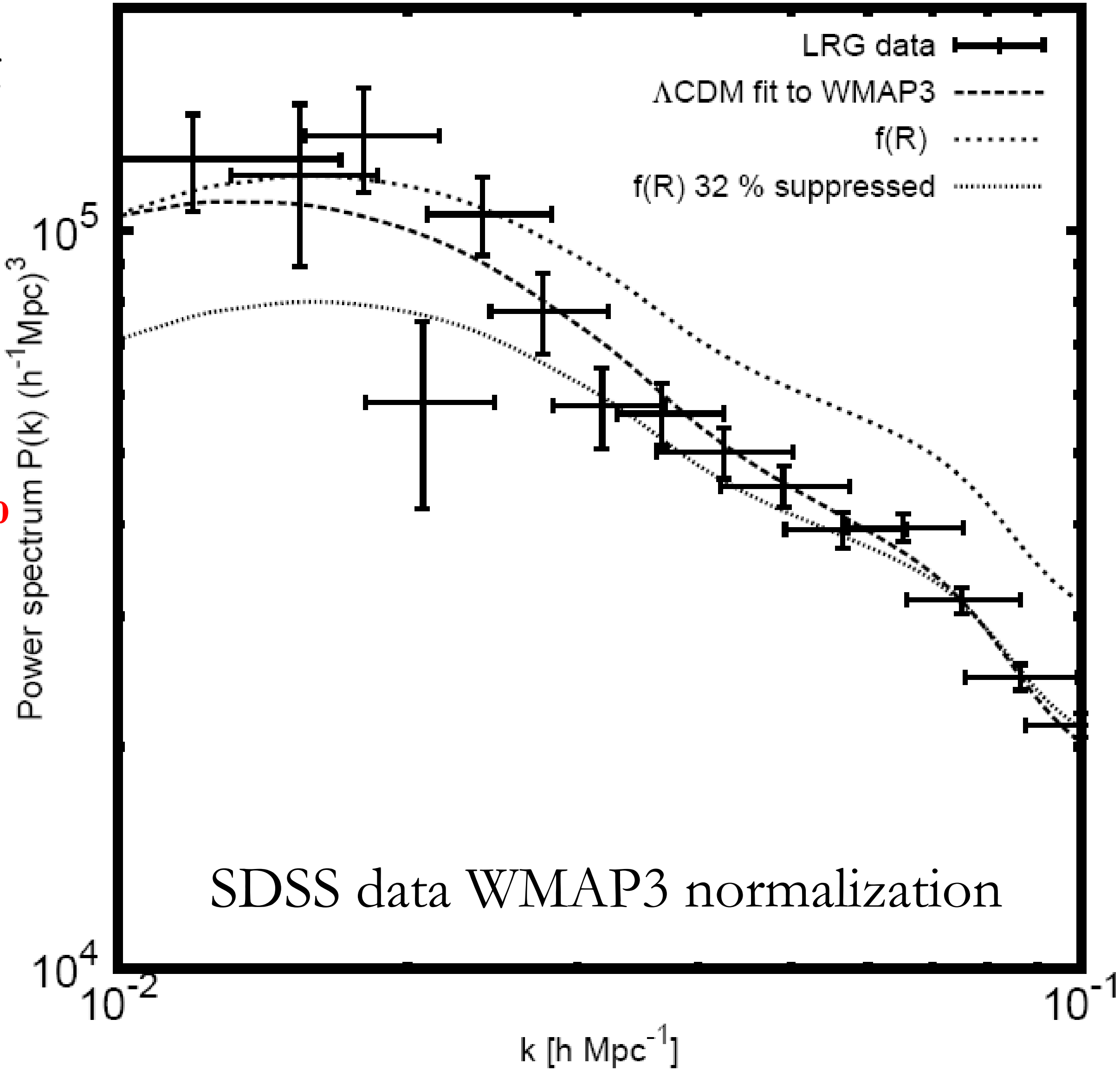
$$f(R) = \alpha R_* \ln\left(1 + \frac{R}{R_*}\right)$$

$$\alpha = 2$$
$$\Omega_M \sim 0.25$$
$$|f_R| \sim 0.2$$

✓ ΛCDM (parameters according to WMAP3) gives excellent fit ($\chi^2 = 11.2$).

✓ *f(R)* fit is **13 σ out**.

✓ Even with the best fit, **32 % supression** is required and fit does not improve (**4.8 σ out**) significantly.



ACD, Dobado & Maroto

PRL 103 : 179001, (2009).

CONCLUSIONS

- A completely general fourth order differential equation for δ have been obtained.
- This expression is independent of the *f(R)* theory and valid for any scale *k*.
- For **EH** with/without cosmological constant actions, well-known results are recovered.
- ***Quasi-static equation was proved to DEPEND on the chosen*** *f(R)* ***and it is NOT always VALID in sub-Hubble modes.***
- ***For any proposed*** *f(R)* ***models, obtained exact equation allows to rule non-viable models out.***

Further details…

- Phys. Rev. Lett. 103: 179001, 2009.

- Phys. Rev. D77: 123515, 2008.