Magnetic confinement and the solar tachocline

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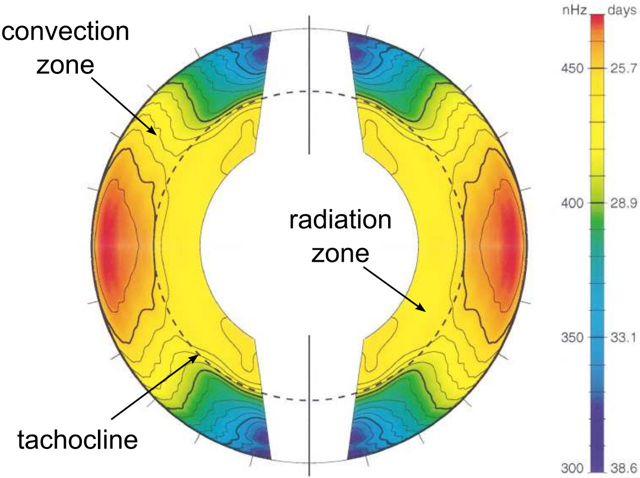
Outline

1. The Sun's meridional flows and magnetic field

2. Does the Sun have a tilted magnetic field?

The solar rotation



Schou et al. 1998

The solar rotation

The solar tachocline

E. A. Spiegel [1] **and J.-P. Zahn** [1,2]

[1] Astronomy Department, Columbia University, New York, NY 10027, USA
[2] Observatoire Midi-Pyrénées, 14 avenue E. Belin, F-31400 Toulouse, France

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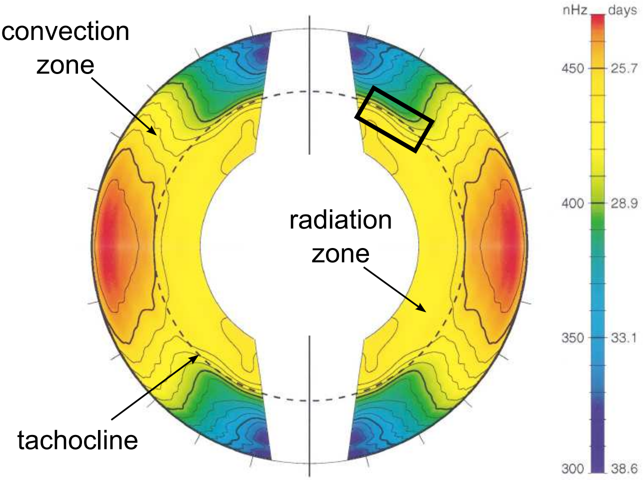
[1] When the existence of such a layer was first adumbrated, the term *tachycline* was proposed (Spiegel 1972). Here we defer to the terminological sensibilities of D.O. Gough and modify that neologism.

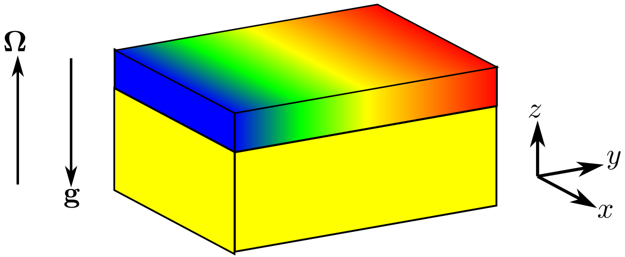
Words matter.

The solar rotation



Schou et al. 1998

Hydrodynamic burrowing



Consider linear, Boussinesq perturbations within stably stratified interior.

$$\frac{\partial \mathbf{u}}{\partial t} + 2\Omega \mathbf{e}_z \times \mathbf{u} = -\nabla \hat{p} + \hat{T}\mathbf{e}_z$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \hat{T}}{\partial t} + N^2 u_z = \kappa \nabla^2 \hat{T}$$

Hydrodynamic burrowing

$$\frac{\partial u_x}{\partial t} - 2\Omega u_y = 0$$
$$\frac{\partial u_y}{\partial t} + 2\Omega u_x = -\frac{\partial \hat{p}}{\partial y}$$
$$\frac{\partial u_z}{\partial t} = -\frac{\partial \hat{p}}{\partial z} + \hat{T}$$
$$\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$
$$\frac{\partial \hat{T}}{\partial t} + N^2 u_z = \kappa \nabla^2 \hat{T}$$

Hydrodynamic burrowing

$$\frac{\partial u_x}{\partial t} - 2\Omega u_y = 0$$

$$\cancel{\frac{\partial u_y}{\partial t}} + 2\Omega u_x = -\frac{\partial \hat{p}}{\partial y} \quad \text{cyclostrophic balance}$$

$$\cancel{\frac{\partial u_z}{\partial t}} = -\frac{\partial \hat{p}}{\partial z} + \hat{T} \quad \text{hydrostatic balance}$$

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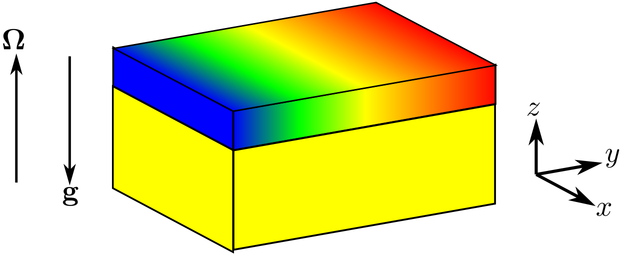
If $\left|\frac{\partial^2}{\partial y^2}\right| = \frac{1}{L^2} \ll \left|\frac{\partial^2}{\partial z^2}\right|$ then

$$\frac{\partial u_x}{\partial t} = -\frac{L^4}{t_{\mathrm{ES}}} \frac{\partial^4 u_x}{\partial z^4} \qquad \text{where } t_{\mathrm{ES}} = \frac{N^2}{4\Omega^2} \frac{L^2}{\kappa}$$

Hydrodynamic burrowing



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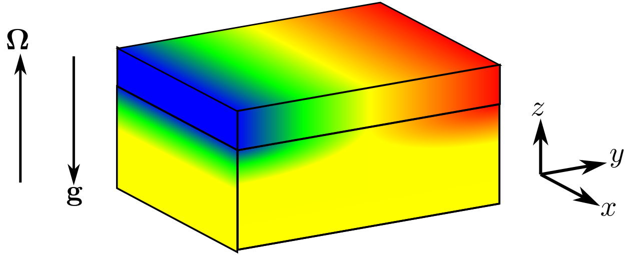

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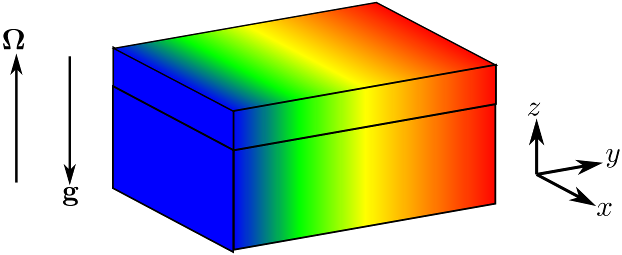
Hydrodynamic burrowing



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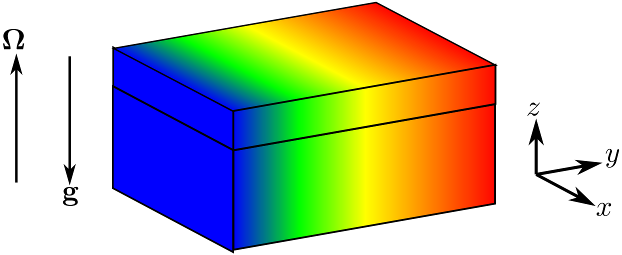
Hydrodynamic burrowing

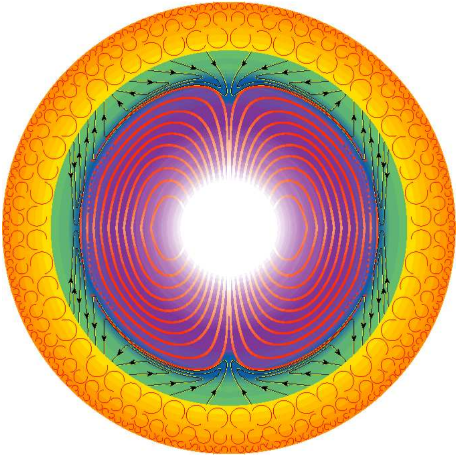


$$\frac{\partial u_x}{\partial t} = -\frac{L^4}{t_{\mathrm{ES}}}\frac{\partial^4 u_x}{\partial z^4}$$

- Angular momentum transported by meridional flows (not viscous diffusion)
- Adding viscosity inhibits the burrowing of meridional flows, but *enhances* the spread of differential rotation

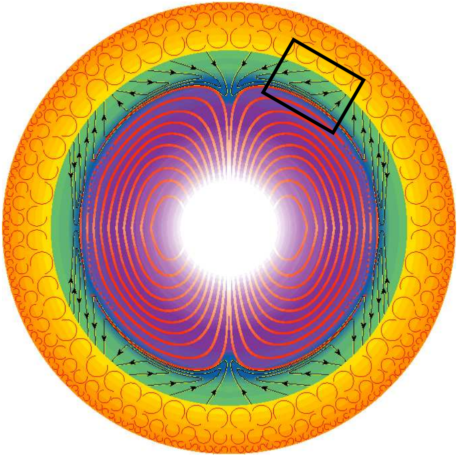
What stops the burrowing?

The interior magnetic field



Gough & McIntyre 1998

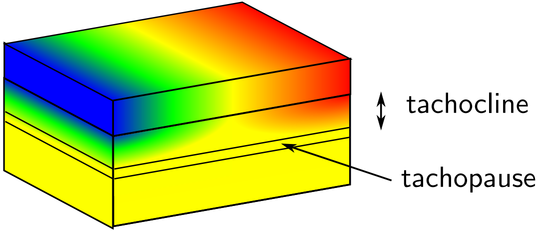
The interior magnetic field



Gough & McIntyre 1998

The Gough & McIntyre model

Add a uniform, horizontal magnetic field into the background.
Can now find steady boundary-layer solutions of the linear equations.



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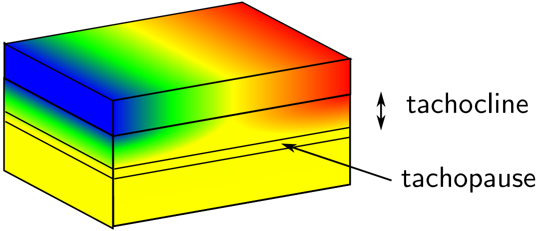
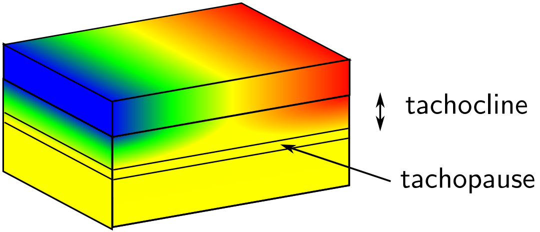


Assume tachocline is (mean) field free.
Estimate u_z from thermal-wind balance and local thermal equilibrium.
Can add "non-linear flavour" by equating tachopause thickness to $\eta/|u_z|$.

Wood, McCaslin & Garaud 2011

Can we couple the tachocline to the tachopause?
(and to the convection zone and radiation zone?)



Wood, McCaslin & Garaud 2011

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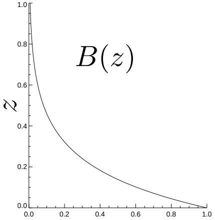
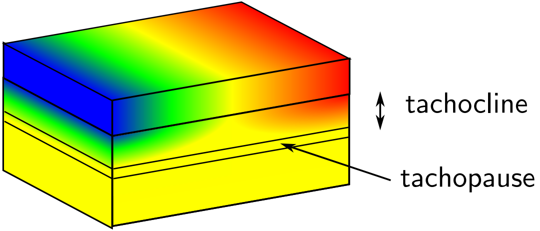


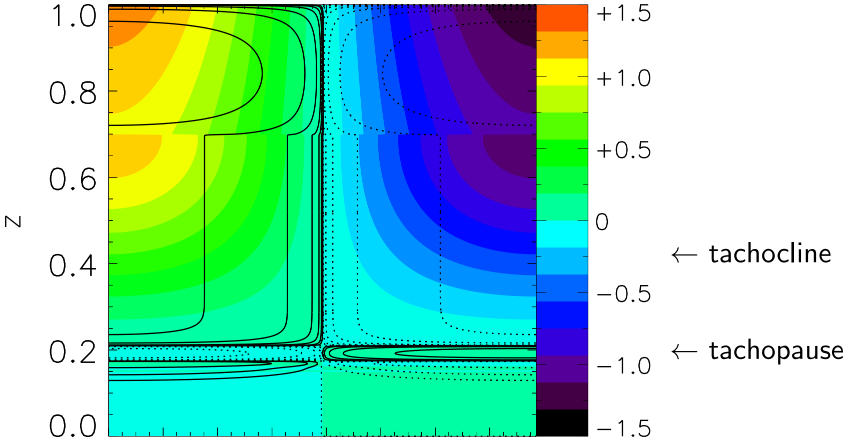


Incorporate uniform downwelling U and confined field B into background.

Add forcing term and turbulent thermal diffusivity in convection zone.

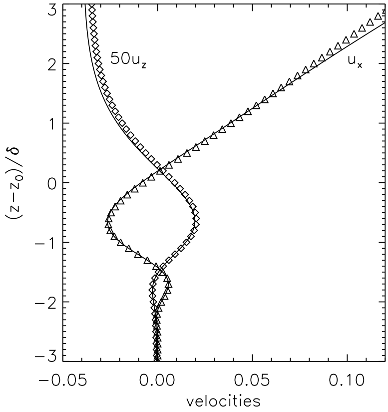
Where does tachopause form? Does u_z follow GM98 scaling?

Wood, McCaslin & Garaud 2011



Wood, McCaslin & Garaud 2011

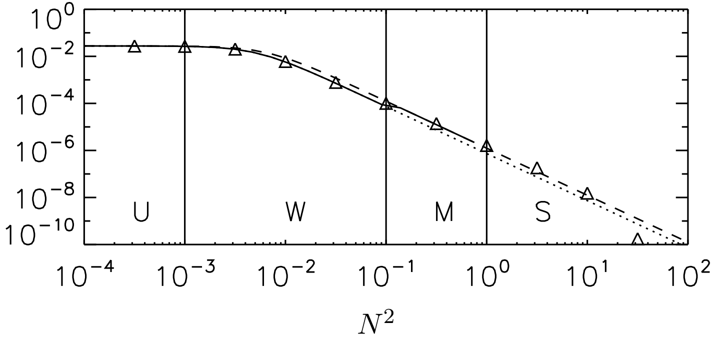
- Tachopause thickness $\delta \sim \eta/U$
- Tachopause forms where $B^2 \sim \Omega\, \eta\, (L/\delta)^2$



Wood, McCaslin & Garaud 2011

- Within the tachocline,

$$u_z \sim \frac{\bar{u}_{\rm cz}}{G_{\rm cz} + G_{\rm tp} + G_{\rm tc}\,\Omega\,t_{\rm ES}}$$



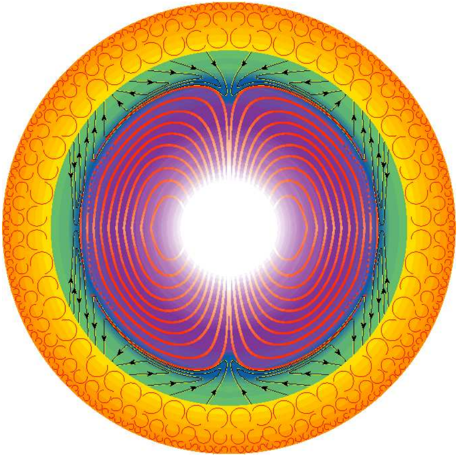
Wood, McCaslin & Garaud 2011

- With realistic parameters, recover GM98 scaling for meridional flow
- But tachopause structure quite different

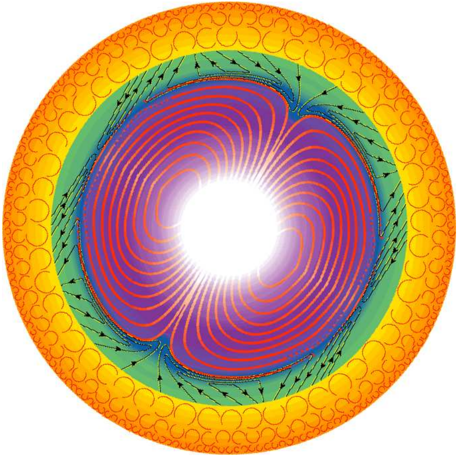
  $\Rightarrow$ different prediction for $|B|$

- Lessons for numerical models

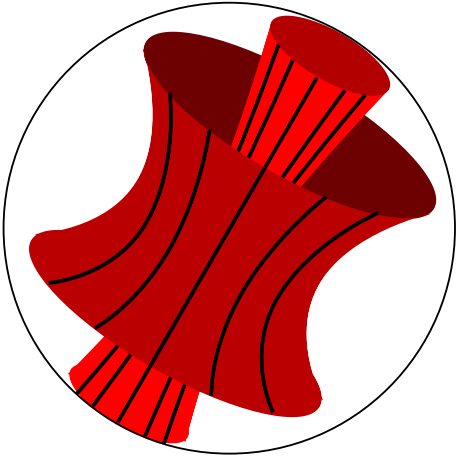
Why an axial dipole?



Why an axial dipole?



A tilted magnetic field



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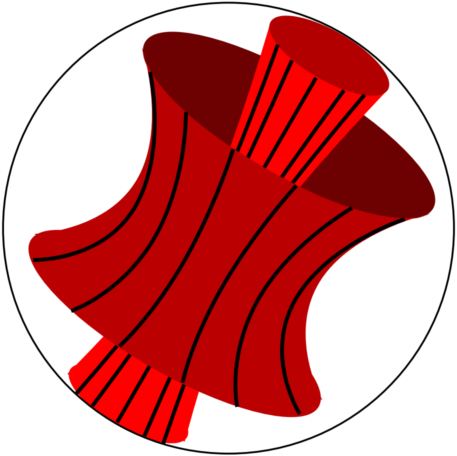

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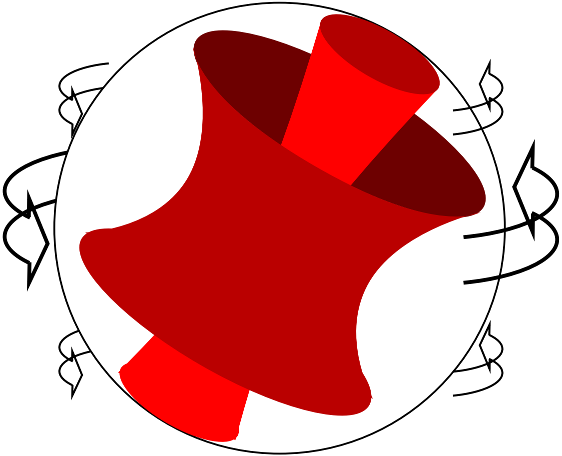


A tilted magnetic field

What happens to the dipole axis?

A tilted magnetic field

What happens to the dipole axis?

1. The dipole aligns with the rotation axis
2. The dipole aligns with the equator
3. Nothing

A tilted magnetic field

What happens to the dipole axis?

1. The dipole aligns with the rotation axis
2. The dipole aligns with the equator
3. Nothing

Angular momentum conservation will tell us!

A tilted magnetic field

$$\mathbf{v}(\mathbf{r},t) = \mathbf{\Omega}(\psi,t) \times \mathbf{r}$$

where

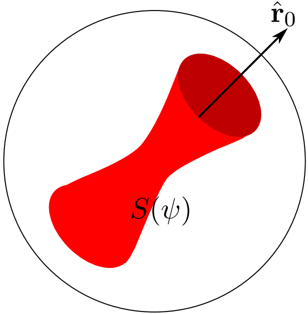
$$\mathbf{\Omega}(\psi,t) = \omega(\psi,t)\,\hat{\mathbf{r}}_0(t) + \mathbf{\Omega}'(t)$$

and

$$\mathbf{\Omega}' \cdot \hat{\mathbf{r}}_0 = 0.$$

Then

$$\frac{\mathrm{d}\hat{\mathbf{r}}_0}{\mathrm{d}t} = \mathbf{\Omega} \times \hat{\mathbf{r}}_0 = \mathbf{\Omega}' \times \hat{\mathbf{r}}_0.$$



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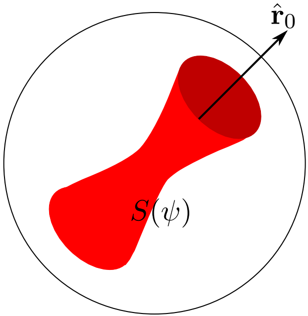

Suppose that $\mathbf{f} = \lambda(\mathbf{v}_{\mathrm{tc}} - \mathbf{v})$, where

$$\mathbf{v}_{\mathrm{tc}}(\mathbf{r}) = \Omega_{\mathrm{tc}}(\theta)\mathbf{e}_z \times \mathbf{r}.$$

A tilted magnetic field

$$\frac{\mathrm{d}\hat{\mathbf{r}}_0}{\mathrm{d}t} = \boldsymbol{\Omega}' \times \hat{\mathbf{r}}_0$$

$$\hat{\mathbf{r}}_0 \times \frac{\mathrm{d}\boldsymbol{\Omega}'}{\mathrm{d}t} = \frac{8\pi\lambda}{3I}\hat{\mathbf{r}}_0 \times \left[\bar{\Omega}_{\mathrm{tc}}\mathbf{e}_z - \boldsymbol{\Omega}'\right] - \langle\omega\rangle_I \, \boldsymbol{\Omega}'$$

$$\frac{\partial\omega}{\partial t} = \ldots$$

where $\bar{\Omega}_{\mathrm{tc}} = \dfrac{\int_{\mathrm{sph.}} \Omega_{\mathrm{tc}} \sin^2\theta \, \mathrm{d}S}{\int_{\mathrm{sph.}} \sin^2\theta \, \mathrm{d}S}$ and $\langle\omega\rangle_I = \dfrac{\int_\psi \omega \, \mathrm{d}I}{\int_\psi \mathrm{d}I}$

A tilted magnetic field

$$\frac{\mathrm{d}\hat{\mathbf{r}}_0}{\mathrm{d}t} = \mathbf{\Omega}' \times \hat{\mathbf{r}}_0$$

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Steady states require that $\langle\omega\rangle_I \, \hat{\mathbf{r}}_0 + \mathbf{\Omega}' \; = \; \bar{\Omega}_{\mathrm{tc}}\mathbf{e}_z.$

An example

Suppose the sphere has constant density, a uniform magnetic field, and no internal friction.
Suppose also that $\Omega_{\mathrm{tc}}(\theta) = a + b\cos^2\theta$.

../movies/solar1.mp4	../movies/anti-solar1.mp4

An example

Suppose the sphere has constant density, a uniform magnetic field, and no internal friction.
Suppose also that $\Omega_{\rm tc}(\theta) = a + b\cos^2\theta$.

`../movies/solar1.mp4`	`../movies/anti-solar1.mp4`

There is a tilted steady state with $\cos\theta_0 = \pm\sqrt{3/5} \ \Rightarrow \theta_0 \approx 39^\circ$.

Stability of the axial dipole

We have

$$\frac{\mathrm{d}\hat{\mathbf{r}}_0}{\mathrm{d}t} = \boldsymbol{\Omega}' \times \hat{\mathbf{r}}_0$$

$$\hat{\mathbf{r}}_0 \times \frac{\mathrm{d}\boldsymbol{\Omega}'}{\mathrm{d}t} = \frac{8\pi\lambda}{3I}\hat{\mathbf{r}}_0 \times \left[\bar{\Omega}_{\mathrm{tc}}\mathbf{e}_z - \boldsymbol{\Omega}'\right] - \langle\omega\rangle_I \boldsymbol{\Omega}'$$

$$\frac{\partial\omega}{\partial t} = \ldots$$

Suppose that

$$\hat{\mathbf{r}}_0 = \mathbf{e}_z + \delta\hat{\mathbf{r}}_0$$

$$\boldsymbol{\Omega}' = 0 + \delta\boldsymbol{\Omega}'$$

$$\omega = \Omega + \delta\omega$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\hat{\mathbf{r}}_0 = \delta\boldsymbol{\Omega}' \times \mathbf{e}_z$$

$$\mathbf{e}_z \times \frac{\mathrm{d}}{\mathrm{d}t}\delta\boldsymbol{\Omega}' = -\frac{8\pi\lambda}{3I}\mathbf{e}_z \times \left[\bar{\Omega}_{\mathrm{tc}}\delta\hat{\mathbf{r}}_0 + \delta\boldsymbol{\Omega}'\right] - \langle\Omega\rangle_I \delta\boldsymbol{\Omega}'$$

Stability of the axial dipole

Growth rates σ are found by solving

$$0 = \sigma^2 \left(\sigma + \frac{8\pi\lambda}{3I} \right)^2 + \left(\sigma \left\langle \Omega \right\rangle_I + \frac{8\pi\lambda}{3I} \bar{\Omega}_{\rm tc} \right)^2 .$$

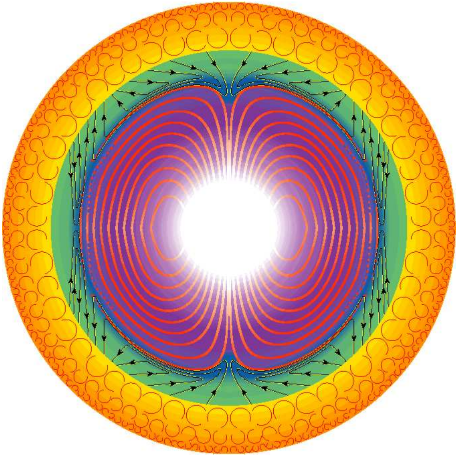
Hence the aligned dipole is stable if and only if

$$\frac{\left\langle \Omega \right\rangle_I}{\bar{\Omega}_{\rm tc}} > 1.$$

Stability of the axial dipole

../movies/tilt-up.mp4

The solution?



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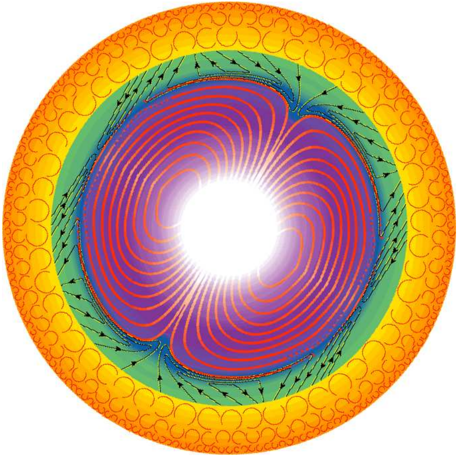


The solution?

Thank you!

The tachopause solution

$$\hat{u}_z = \mathrm{i}k\delta u_{\mathrm{t}} \left[I_1(\zeta) - \tfrac{\pi}{2} \operatorname{Re}\left\{ \exp\left(-\tfrac{1+\mathrm{i}}{\sqrt{2}}\zeta \right) \right\} \right] + \tfrac{1}{2}k^2\delta^2 T_{\mathrm{t}} \left[I_2(\zeta) - \tfrac{\pi}{2} \operatorname{Re}\left\{ (\gamma - \mathrm{i}\tfrac{\pi}{4}) \exp\left(-\tfrac{1+\mathrm{i}}{\sqrt{2}}\zeta \right) \right\} \right]$$

where γ is the Euler–Mascheroni constant, $\gamma = 0.577...$, $\zeta = \exp((z_0 - z)/\delta)$, and

$$I_1(\zeta) = \int_0^\infty \frac{s\,\mathrm{d}s}{\mathrm{e}^{\zeta s}} \frac{1}{1+s^4}\,,$$
$$I_2(\zeta) = \int_0^\infty \frac{s\,\mathrm{d}s}{\mathrm{e}^{\zeta s}} \frac{\gamma + \ln s}{1+s^4}\,.$$

Meridional flow in the tachocline

$$w_{\rm t} = \frac{-{\rm i}\bar{u}_{\rm cz}}{\frac{kd}{2\tau_c}\coth\left(\frac{1-h}{d}\right) + \frac{4}{\pi k\delta} + \frac{n_{\rm rz}^2}{2k^2{\rm E}_\kappa}[G_1 + G_2 - G_3]}$$

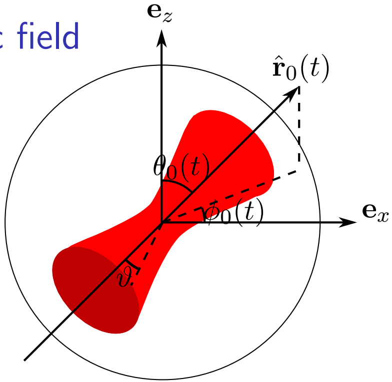
where

$$G_1 = kD - \frac{f_0[2 - 2\,{\rm sech}\,kD + \tanh kz_0 \tanh kD] + \tanh k(1-h)\tanh kD}{f_0[\tanh kz_0 + \tanh kD] + \tanh k(1-h)[\tanh kz_0 \tanh kD + 1]}$$

$$G_2 = \frac{kd}{k^2d^2 - 1}\left[kd - \frac{\tanh k(1-h)}{\tanh(\frac{1-h}{d})}\right]\frac{1 - {\rm sech}\,kh\cosh kz_0}{f_0\tanh kh + \tanh k(1-h)}$$

$$G_3 = \gamma k\delta \frac{f_0[1 - {\rm sech}\,kD]\tanh kz_0 + \tanh k(1-h)\tanh kD\tanh kz_0}{f_0[\tanh kz_0 + \tanh kD] + \tanh k(1-h)[\tanh kz_0\tanh kD + 1]}$$

A tilted magnetic field



$$\frac{\mathrm{d}I}{\mathrm{d}\psi}\frac{\partial\omega}{\partial t} = \frac{\partial}{\partial\psi}\left(\mu\frac{\partial\omega}{\partial\psi}\right) + \frac{\mathrm{d}A}{\mathrm{d}\psi}\lambda\sin^2\vartheta\left(\left[\Omega_{\mathrm{cz}}(\theta)\frac{\partial\cos\theta}{\partial\cos\vartheta}\right]_{C(\psi)} - \omega\right)$$