Classification of eigenmodes of spherically symmetric stars

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Introduction

- It is a fundamental problem in the theory of stellar pulsation to examine the reaction of a star to a small perturbation.
- We discuss (adiabatic) eigenmodes of spherically symmetric stars. Particularly, we pay attention to what kinds of eigenmodes exist and how they are classified mathematically.

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Local waves in a star

In a non-rotating and non-magnetic star in a hydrostatic equilibrium, the pressure gradient and the gravity are in balance, each of which can be a restoring force of local waves.

restoring force	type of waves	frequency
pressure	acoustic waves	high
buoyancy	(internal) gravity waves	low

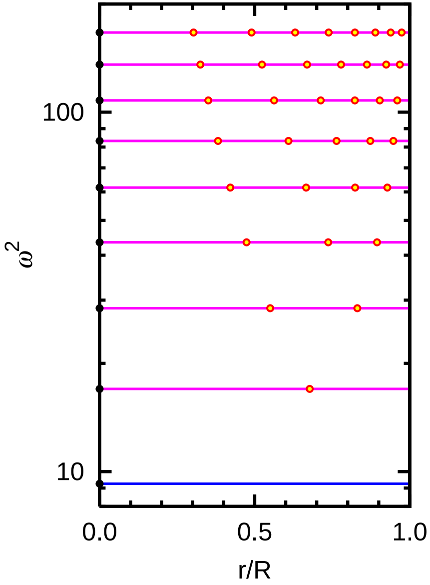
These two types of local waves are the main constituents of the global eigenmodes.

Radial modes

- Constituent: acoustic waves
- ***Sturm–Liouville problem*** (with singular endpoints)
    - infinite number of discrete eigenvalues
    - existence of the minimum eigenvalue (ω_0^2)
    - eigenvalues are not bounded from above
    - the eigenfunction that corresponds to the n-th eigenvalue (ω_n^2) has n nodes

    ⋮

Polytropic model with index 3



Nonradial modes under the Cowling approximation (1)

- Buoyancy participates in the problem.
- A ***fourth order*** system of ordinary differential equations
- ***Cowling approximation*** [Cowling (1941)]:

  neglect of the perturbation to the gravitational field,

  which is good
  - in the low density regions (envelope and atmosphere)
  - for short wavelength oscillations

Nonradial modes under the Cowling approximation (2)

Once we accept the Cowling approximation, the problem is simplified significantly.

- A ***second order*** system of ordinary differential equations
- ***Sturm–Liouville problems*** both in the high and low frequency limits.
- Eigenmodes are classified into three categories:
  - p modes (high frequency branch)
  - g modes (low frequency branch)
  - f modes (a single intermediate mode for a given spherical degree) (only when the spherical degree $\ell \geq 2$)

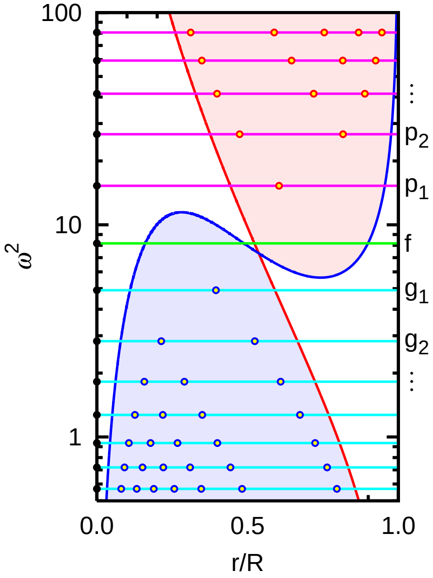
Nonradial modes under the Cowling approximation (3)

For models with ***mild*** central condensation,

modes	constituents
p	acoustic waves
g	(internal) gravity waves
f	surface gravity waves

- the eigenfunction (the radial displacement) of the n-th p (g) mode has n nodes, whereas that of the f mode has no node.

Polytropic model with index 3, $\ell = 2$



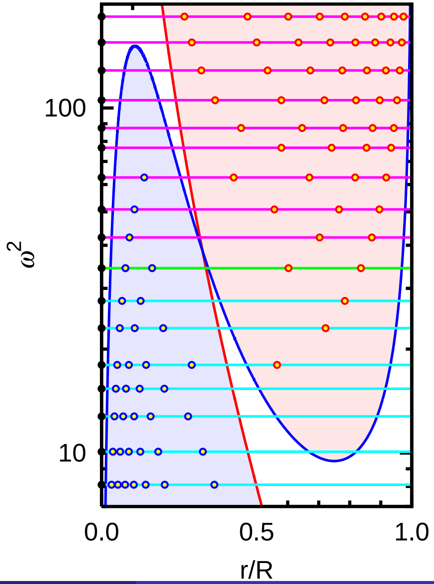
Nonradial modes under the Cowling approximation (4)

Polytropic model with index 4, $\ell = 2$

There is a problem.

In models with ***high*** central mass concentration,

- Some low order modes have a gravity-wave character in the core and an acoustic wave character in the envelope (mixed modes).
- No simple relation between the number of nodes of eigenfunctions and the order of eigenmodes.

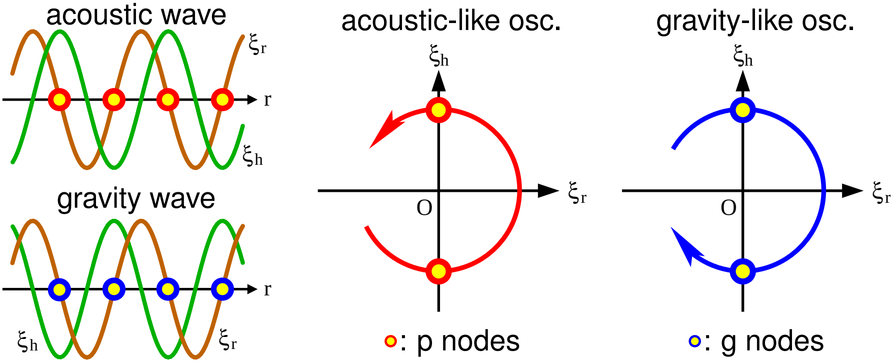


Nonradial modes under the Cowling approximation (5)

Resolution: ***Eckart–Scuflaire–Osaki (ESO) scheme***

(Eckart 1960; Scuflaire 1974; Osaki 1975)

- A pair of components of eigenfunctions, (ξ_r, ξ_h) (the radial and horizontal displacement).
- Examine the trajectory of the point (ξ_r, ξ_h) as a function of the radial distance from the centre (***phase diagram***).



Nonradial modes under the Cowling approximation (6)

ESO scheme (continued)

- Each eigenmode is discriminated by the ***total net rotation angle*** on the phase diagram.
- A practical way of labelling each eigenmode:

n(mode index) = [number of p nodes] - [number of g nodes]

- Mode classification
  - $n > 0$: p_n mode
  - $n < 0$: $\mathrm{g}_{|n|}$ mode
  - $n = 0$: f mode

Dipole modes (1)

There are still problems.

- The Cowling approximation results in the ***dipolar f mode with a non-zero frequency***.
- The ESO scheme has a problem, when it is applied to dipole modes of stellar models with high central mass concentration.
  (Lee 1985; Guenther 1991; Christensen-Dalsgaard & Mullan 1994)

These are simply because the Cowling approximation is not good for low order dipole modes (e.g. Christensen-Dalsgaard & Gough 2001).

Dipole modes (2)

Special treatment of dipole modes (Takata 2005, 2006)

- A specific integral that comes from momentum conservation.
- A second order ordinary differential equation without the Cowling approximation.
- By suitably choosing the components of eigenfunctions, we may construct a specific scheme of dipole-mode classification.

Higher degree modes (without the Cowling approximation) (1)

An unresolved problem

- Practically speaking, the ESO scheme is sufficient for mode classification, but...
  - Admitting the Cowling approximation means throwing away the Poisson equation of the self-gravity.
  - Better insights into the full problem must be valuable, particularly when we interpret highly precise observational results obtained by MOST, COROT, Kepler etc.

Higher degree modes (without the Cowling approximation) (2)

Why is the problem so hard?

- The fourth order (linear) system of ordinary differential equations is much less tractable.
- A possibility of ***accidental degeneracy*** between eigenmodes with the same spherical degree (although any particular example has not been found so far).

Summary

- Radial oscillation is well understood, because it is a classical Sturm–Liouville problem (with singular endpoints).
- The Cowling approximation significantly simplifies the analysis of nonradial modes. The scheme of mode classification based on this (ESO scheme) works well, except for low order dipole modes of equilibrium models with high central mass concentration.
- The problem of dipole-mode oscillations is resolved by casting the differential equations of the full problem into a form to which Eckart's analysis can be applied.
- The classification of higher degree modes without the Cowling approximation is still an unresolved problem.