DOUBLE-DIFFUSIVE BEHAVIOUR AT HIGH AND LOW PRANDTL NUMBER

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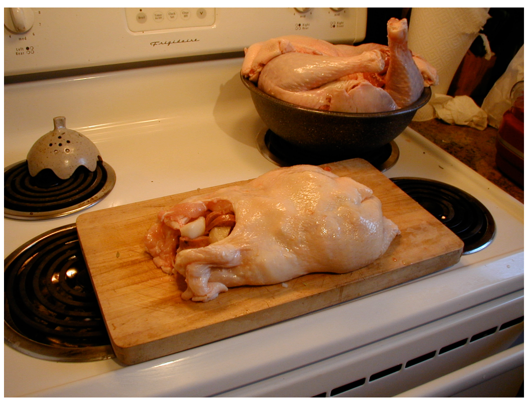
Foreword

"Science is a sport"

(and Douglas likes to win)…



Thanksgiving, 2008

Douglas' reply…

" I was in Cambridge at the beginning of February, and, inspired by your pertinent e-mails, cooked a roti sous (relative to the original) pareil on my birthday.

I had fewer guests than I presume you did, so I cooked much less meat: merely a duck stuffed with a guinnea fowl stuffed with a pheasant and two pigeons (the pigeons there because further in they would have overpowered the adjacent birds - in the original pigeons were avoided), stuffed with a poussin stuffed with a partridge stuffed with two quail, each stuffed with egg (I don't like olives).

It took hours and hours to bone them all, although after I had done it I realized that a much lower standard of integrity would have been acceptable for the inner birds. The dish was appreciated, and the remains kept me fed for almost the rest of the week "

March, 2009.

Foreword

- This talk presents a pot-pourrit of recent exciting results on double-diffusive systems.
  - Transport by small-scale fingering convection and oscillatory convection
  - The formation of thermo-compositional staircases and the excitation of gravity waves by secondary double-diffusive instabilities.

- Following in my mentor's footsteps, I will *attempt* to give a 1-hour talk in 25 minutes…

  … and probably fail.

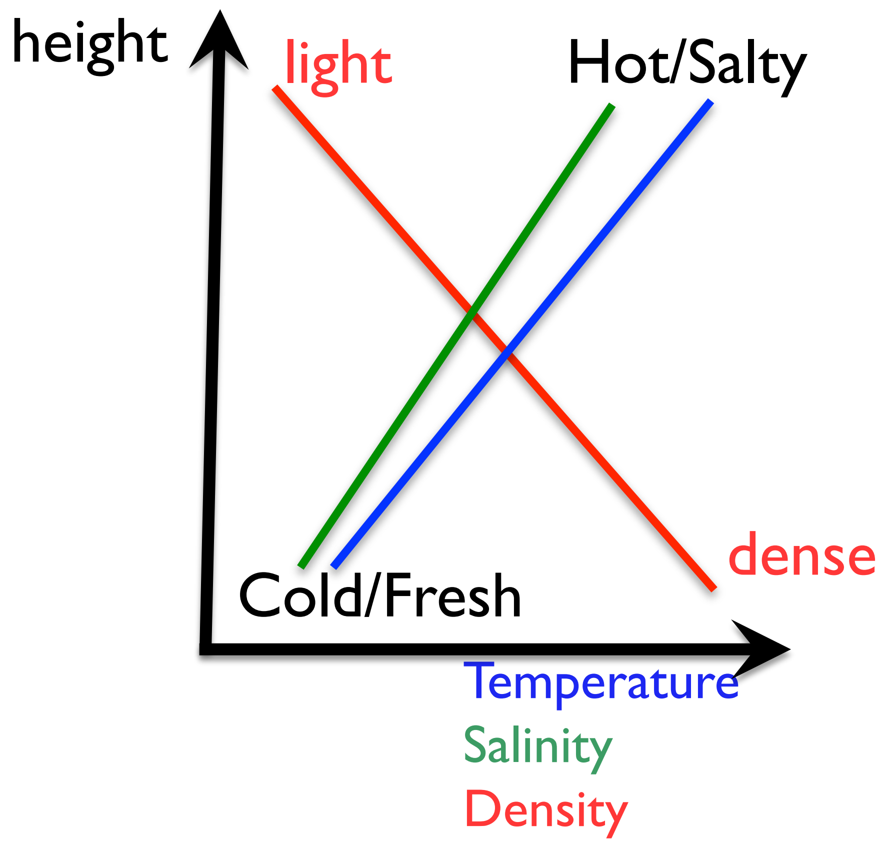
Double-diffusive instabilities

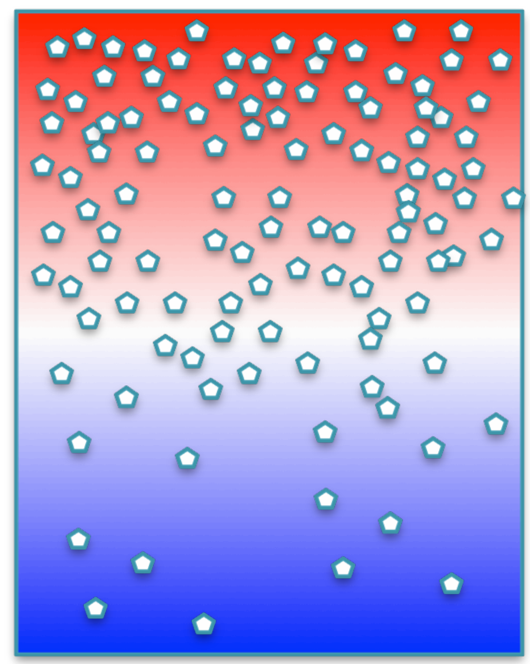
- The fingering instability
- The oscillatory instability
- Mathematical model and linear stability
- Outstanding questions

The fingering instability

- Example of the tropical ocean, Mediterranean:

$$\overline{T}_z, \overline{S}_z > 0 \text{ and } \overline{\rho}_z < 0$$

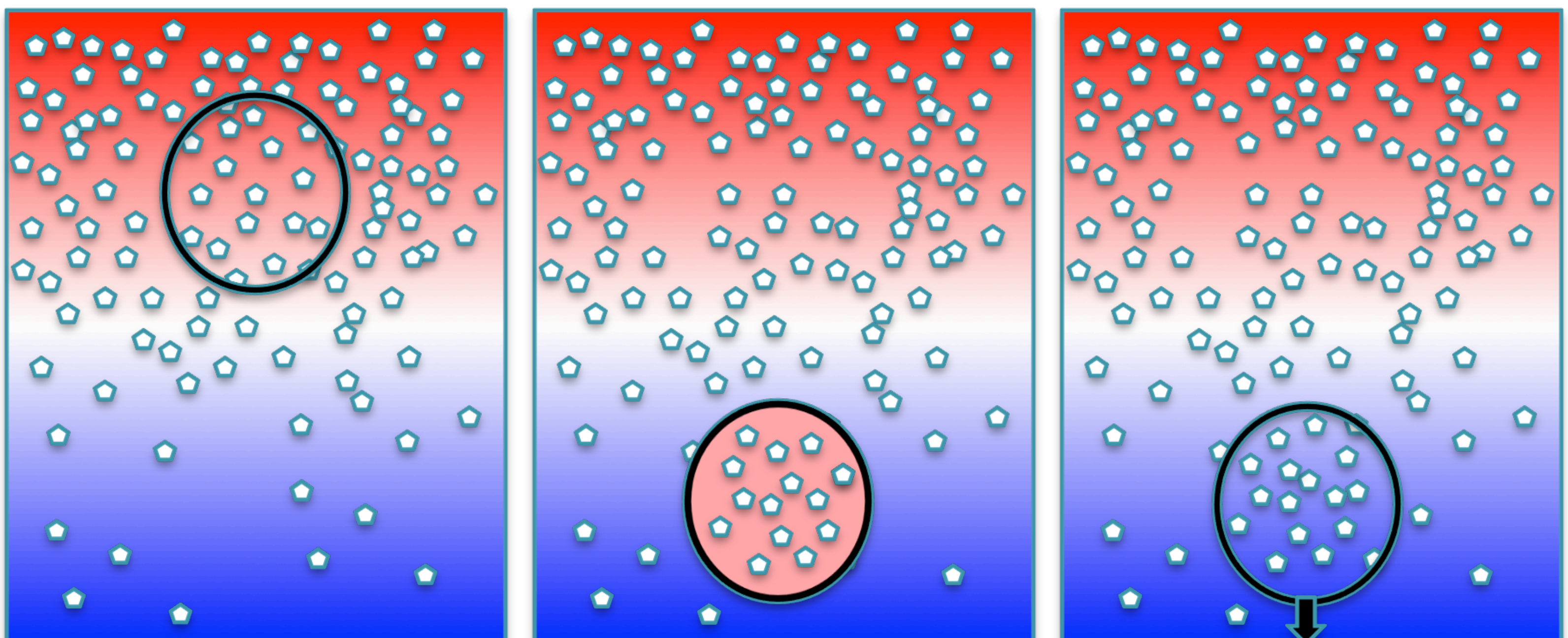




The fingering instability

- Example of the tropical ocean: the fingering instability.

$$\overline{T}_z, \overline{S}_z > 0 \text{ and } \overline{\rho}_z < 0$$



Buoyancy force

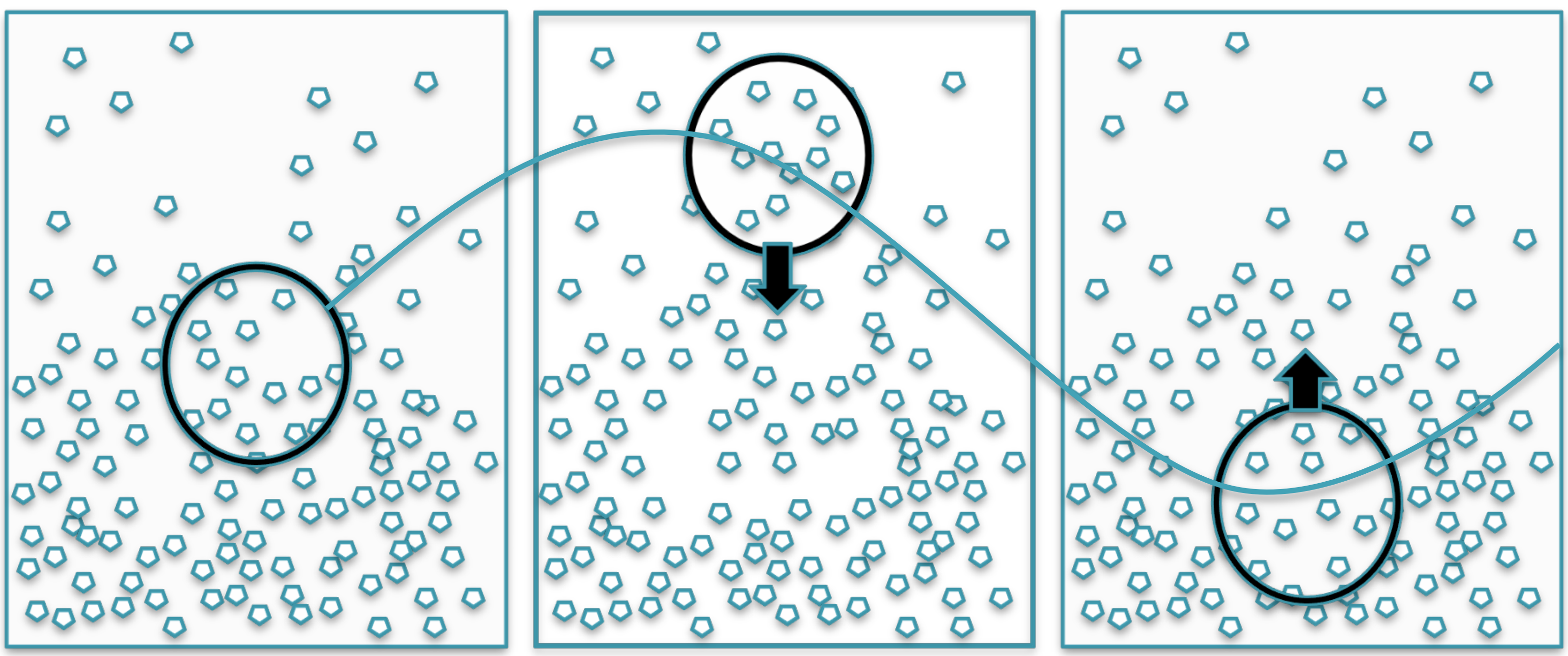
Double-diffusive instabilities

- The fingering instability
- The oscillatory instability
- Mathematical model and linear stability
- Outstanding questions

The oscillatory instability

- Consider an isothermal fluid with stable compositional gradient.

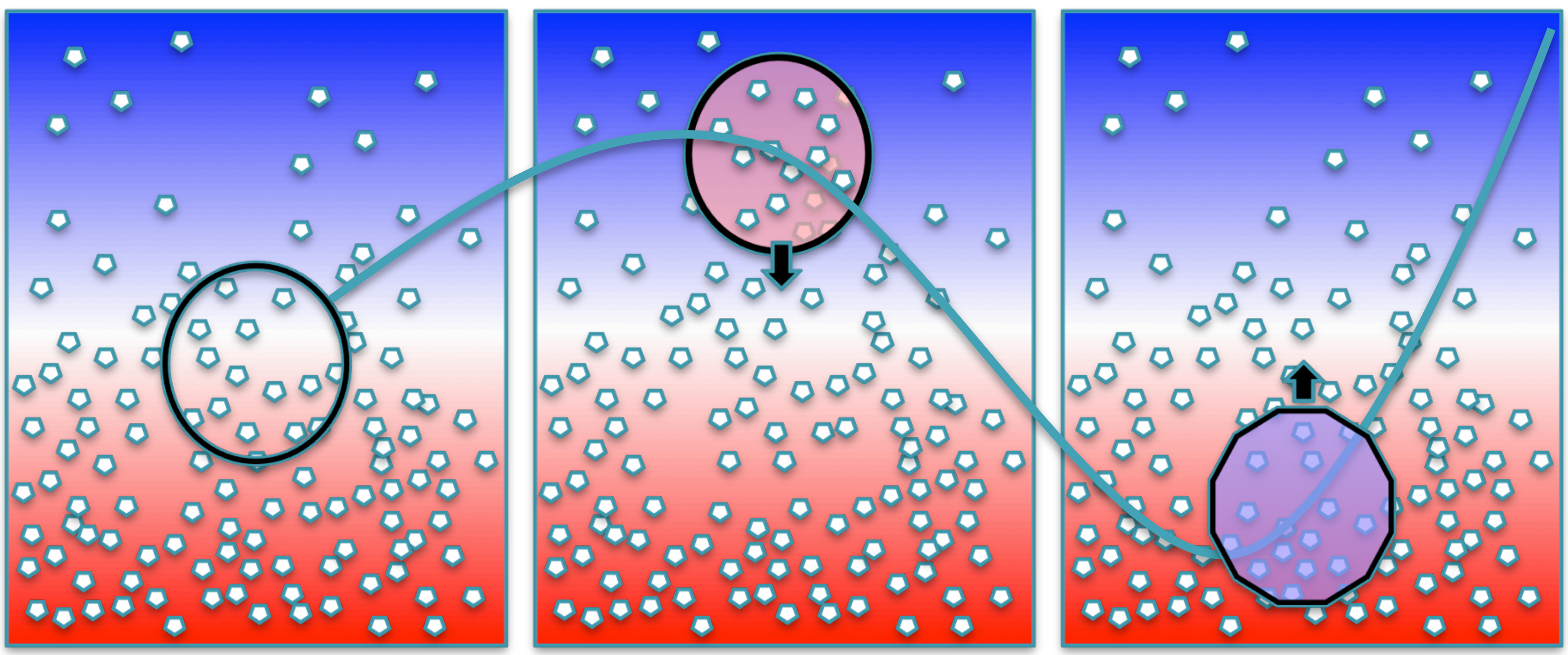
$$\overline{T}_z = 0,\ \overline{S}_z < 0 \text{ and } \overline{\rho}_z < 0$$



The oscillatory instability

- Now assume an unstable temperature gradient (cf. polar ocean): the oscillatory instability.

$$\overline{T}_z,\ \overline{S}_z < 0 \text{ and } \overline{\rho}_z < 0$$



Regimes of double-diffusive convection

- Summary: Double-diffusive instabilities occur when density depends on 2 components, which diffuse at different rates.

  - **Fingering regime (thermohaline)**: rapidly diffusing component (temperature) is stably stratified, slowly diffusing one unstable

    Example in astrophysics: accretion of metal-rich material planet falling onto host star. Entropy gradient stable, mu-gradient unstable.

    Direct instability, long tall finger-like plumes.

  - **Oscillatory regime (semiconvection)**: rapidly diffusing component is unstably stratified, slowly diffusing one stable

    Example in astrophysics: at edge of core-convective stars (cf. semi-convection). Entropy gradient unstable, mu-gradient stable.

    Oscillatory instability, overstable gravity waves.

Double-diffusive instabilities

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Mathematical model

- Governing equations (Boussinesq approximation):

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{\nabla p}{\rho_0} - \frac{\rho}{\rho_0} g \hat{e}_z + \nu \nabla^2 u$$

$$\frac{\partial T}{\partial t} + u \cdot \nabla T = \kappa_T \nabla^2 T$$

$$\frac{\partial S}{\partial t} + u \cdot \nabla S = \kappa_S \nabla^2 S$$

$$\nabla \cdot u = 0$$

$$\frac{\rho}{\rho_0} = -\alpha T + \beta S$$

Mathematical model

Goal: to study double-diffusive instability "in the field".

- Double-diffusive convection scale much, much smaller than system scale.
- Model considered here:
  - Assume **background** temperature and concentration profiles are linear (constant gradients T_{0z}, S_{0z})
  - Assume that all **perturbations** are triply-periodic in domain (L_x,L_y,L_z):

$$q(x,y,z,t) = q(x+L_x,y,z,t)$$
$$= q(x,y+L_y,z,t) = q(x,y,z+L_z,t)$$

  - As a result,

$$T(x,y,z,t) = L_z T_{0z} + T(x,y,z+L_z,t)$$

Mathematical model

- Governing non-dimensional equations (+ fingering case and – oscillatory case):

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = -\nabla p + (T - S)\mathbf{e_z} + \nabla^2 u$$

$$\frac{\partial T}{\partial t} + u \cdot \nabla T \pm w = \nabla^2 T$$

$$\frac{\partial S}{\partial t} + u \cdot \nabla S \pm \frac{w}{R_0} = \tau \nabla^2 S$$

$$\nabla \cdot u = 0$$

$$\mathrm{Pr} = \frac{\nu}{\kappa_T}, \quad \tau = \frac{\kappa_S}{\kappa_T}$$

$$R_0 = \frac{\alpha T_{0z}}{\beta S_{0z}}$$

$$[l] = d = \left(\frac{\kappa_T \nu}{\alpha g |T_{0z}|}\right)^{1/4}, \quad [t] = \frac{d^2}{\kappa_T}, \quad [T] = d|T_{0z}|, \quad [S] = \frac{\alpha}{\beta} d|T_{0z}|$$

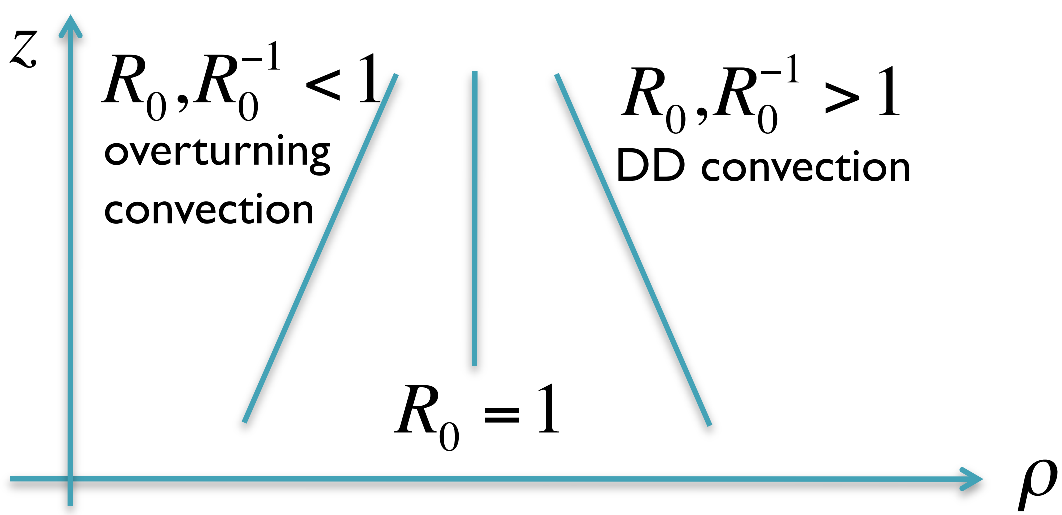
Mathematical model

- The dynamics of double-diffusive instabilities depends principally on the non-dimensional **density ratio:**
- Fingering:

$$R_0 = \frac{\alpha T_{0z}}{\beta S_{0z}} = \frac{\text{Stabilizing temperature stratification}}{\text{Destabilizing salinity stratification}}$$

- Oscillatory:

$$R_0^{-1} = \frac{\beta S_{0z}}{\alpha T_{0z}} = \frac{\text{Stabilizing salinity stratification}}{\text{Destabilizing temperature stratification}}$$



Mathematical model

- Linear stability analysis:
  - Normal mode solution

$$q = \hat{q} e^{ik_x x + ik_y y + ik_z z + st}$$

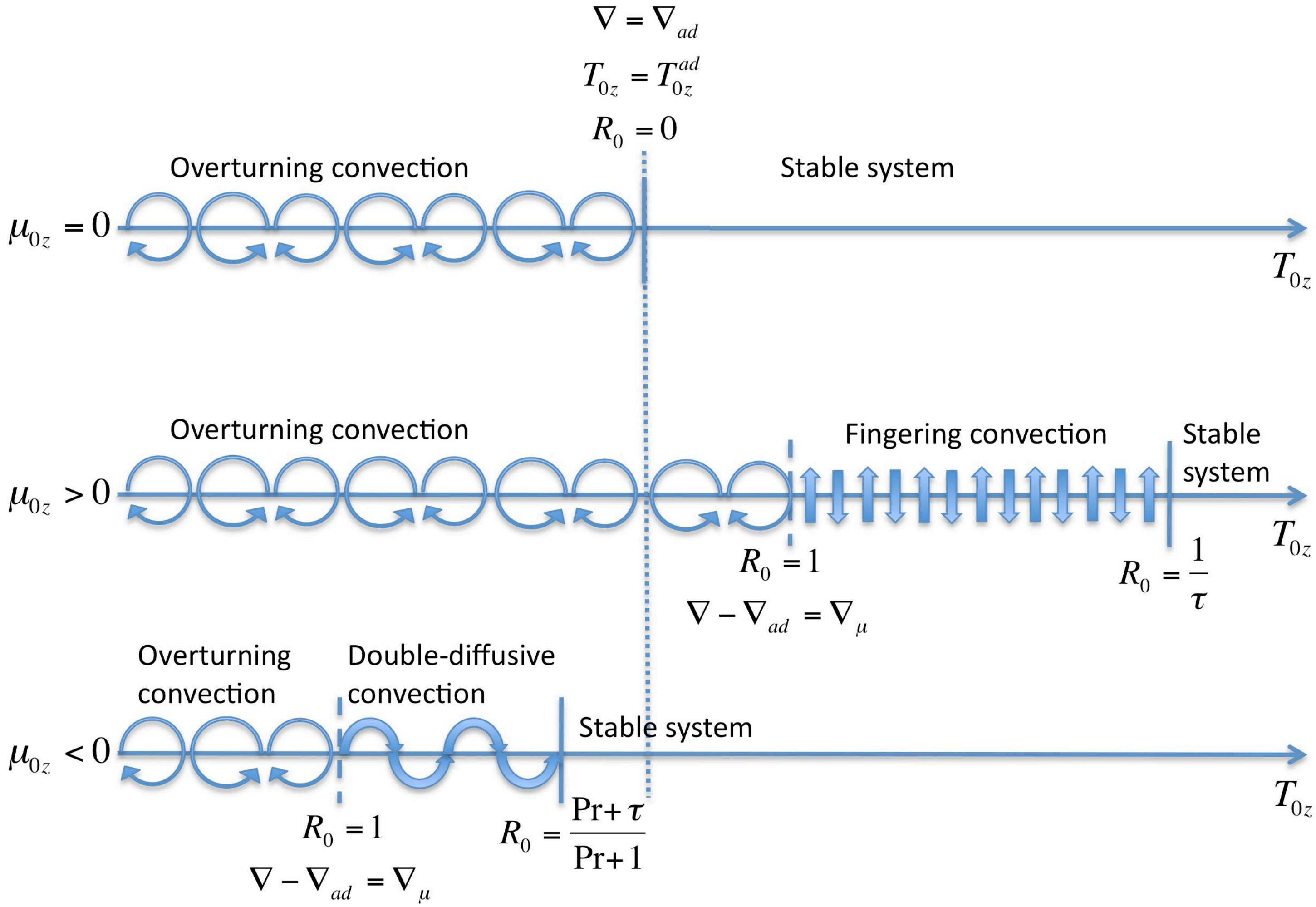
  - Growth rate satisfies a cubic equation in s which depends on **k**, R_ρ, Pr and τ,
  - Fastest growing modes are "elevator modes" ($k_z = 0$)
  - Instability only occurs for

$$1 < R_0 < \frac{1}{\tau} \quad \text{in fingering case}$$

$$1 < R_0^{-1} < \frac{\Pr + 1}{\Pr + \tau} \quad \text{in oscillatory case}$$

Summary

Instability regimes as a function of governing parameters:



Double-diffusive instabilities

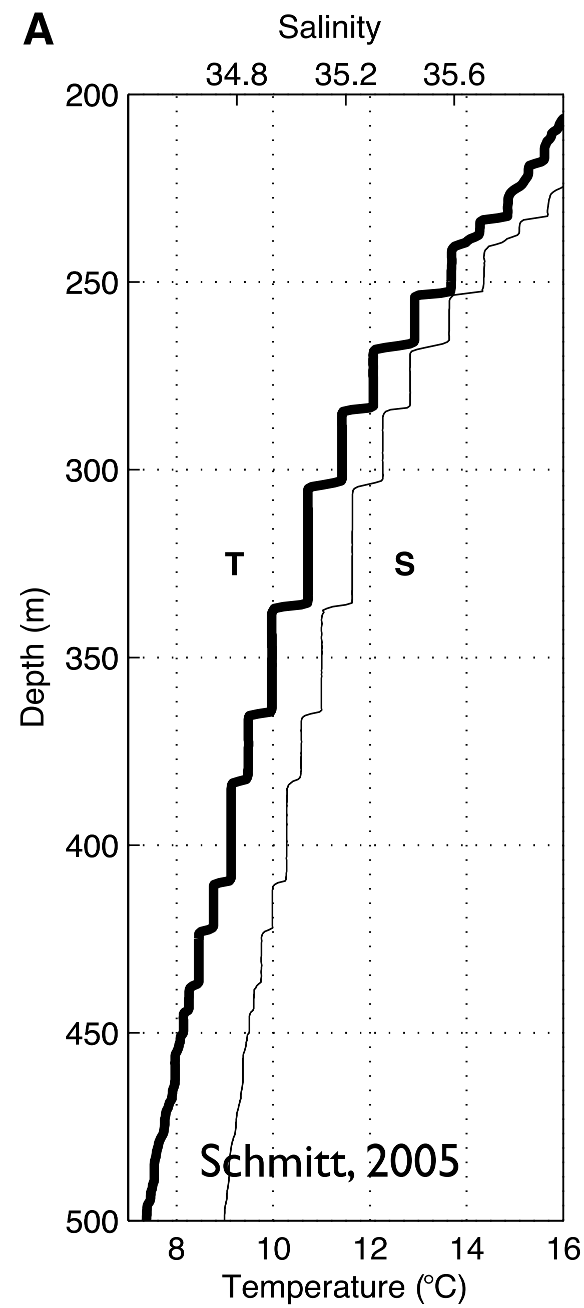
- The fingering instability
- The oscillatory instability
- Mathematical model and linear stability
- Outstanding questions

Outstanding questions

- What are the transport properties of double-diffusive convection in various regimes, as function of the background parameters & fluid properties?
- Can we explain and predict the large-scale dynamics observed to be associated with double-diffusive convection, and in particular large-scale gravity waves, and the formation of **thermocompositional staircases**?
- Can we understand how transport is modified in the presence of large-scale structures?

Thermohaline staircases

- Thermohaline staircases are often observed in ocean thermocline with active fingering convection
  - Layers are typically 10m – 100m deep
  - Can have large horizontal extent (hundreds of kilometers)
  - Individual layers persist for months or more
  - Transport through staircase much larger than through standard fingering convection
- Similar staircases are observed in the polar ocean



Schmitt, 2005

Outstanding questions

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- Can we explain and predict the large-scale dynamics observed to be associated with double-diffusive convection, and in particular large-scale gravity waves, and the formation of **thermocompositional staircases**?
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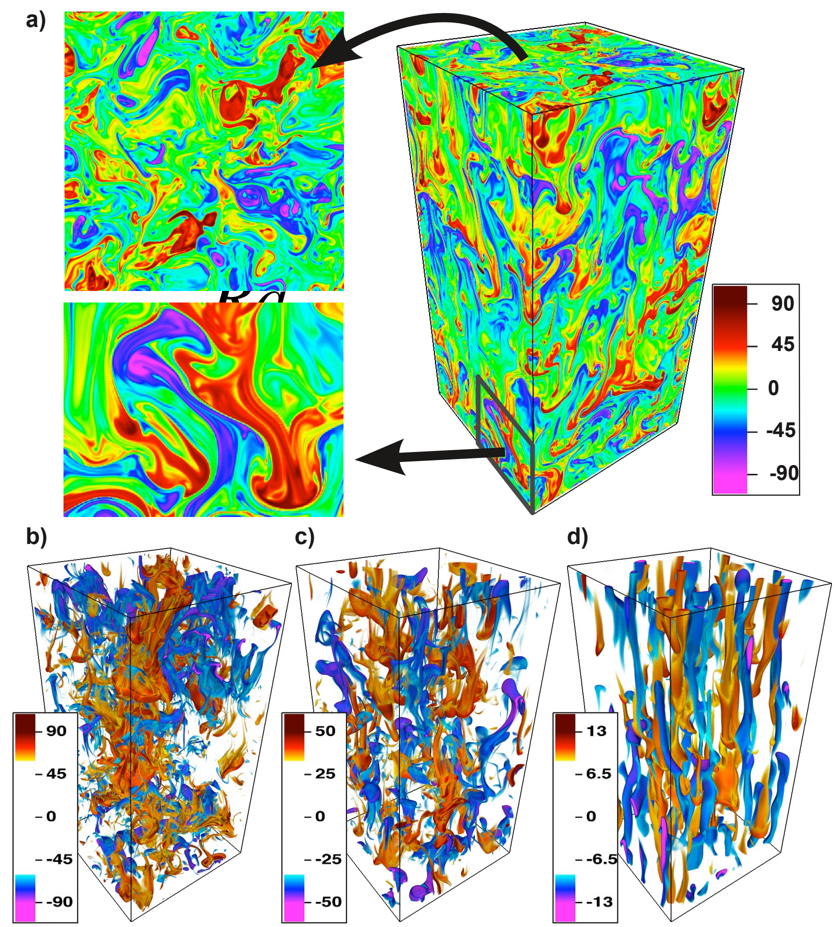
Outstanding questions

- Transport by small-scale instabilities
- The formation of large-scale structures
- Transport through a staircase

Numerical experiments

- Stephan Stellmach developed high-performance 3D code to study double-diffusive convection
- Code is pseudo-spectral, triply periodic, fully resolved on all scales.



$$\mathrm{Pr} = 7$$

$$\tau = 0.01$$

Numerical experiments

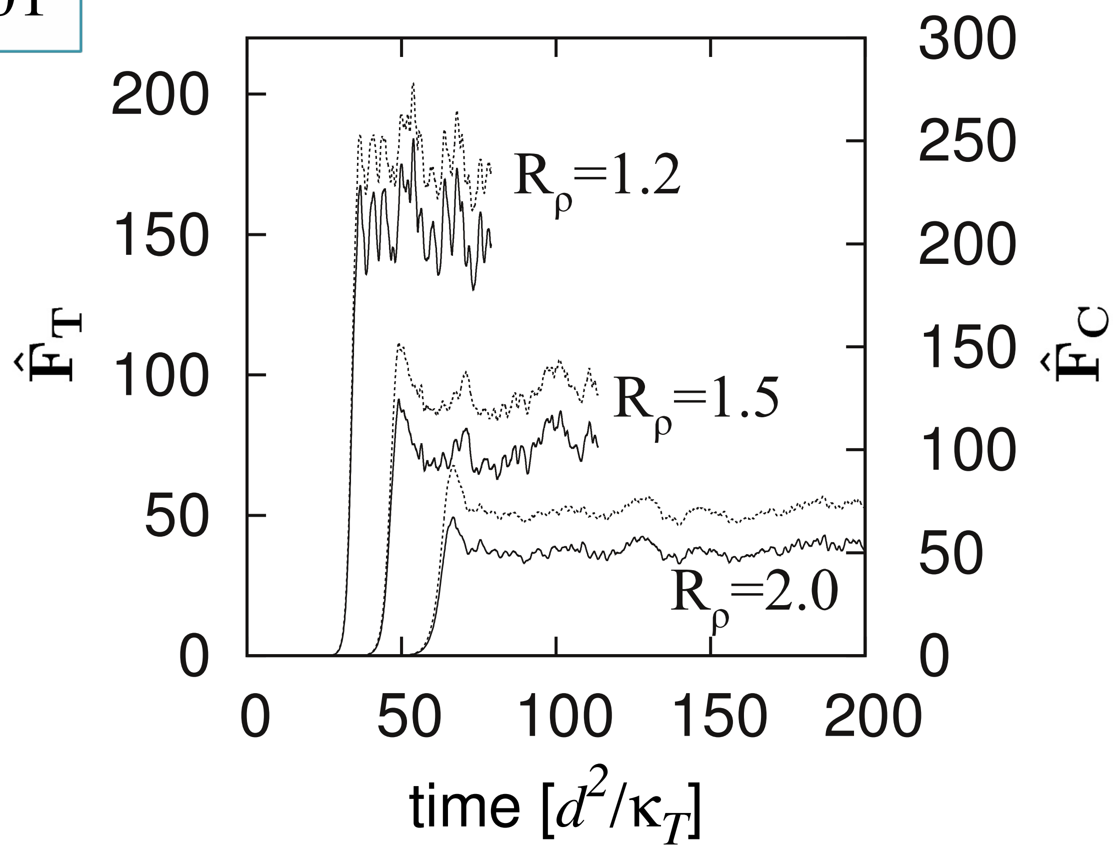
- We define non-dimensional fluxes:
  - The Nusselt number $Nu = \dfrac{\text{Total heat flux}}{\text{Diffused heat flux}}$
  - The total flux ratio $\gamma_{tot} = \dfrac{\text{Total buoy. flux from heat}}{\text{Total buoy. flux from salt}}$

- These quantities are functions of Pr, τ and R_0 and can be measured from "small box" simulations.
  - Box size chosen to contain about 5 x 5 x 10 unstable modes
  - Basic instability grows rapidly, and saturate into homogeneous, statistically steady state.
  - Transport properties are measured.

Numerical experiments

$\mathrm{Pr} = 7$

$\tau = 0.01$

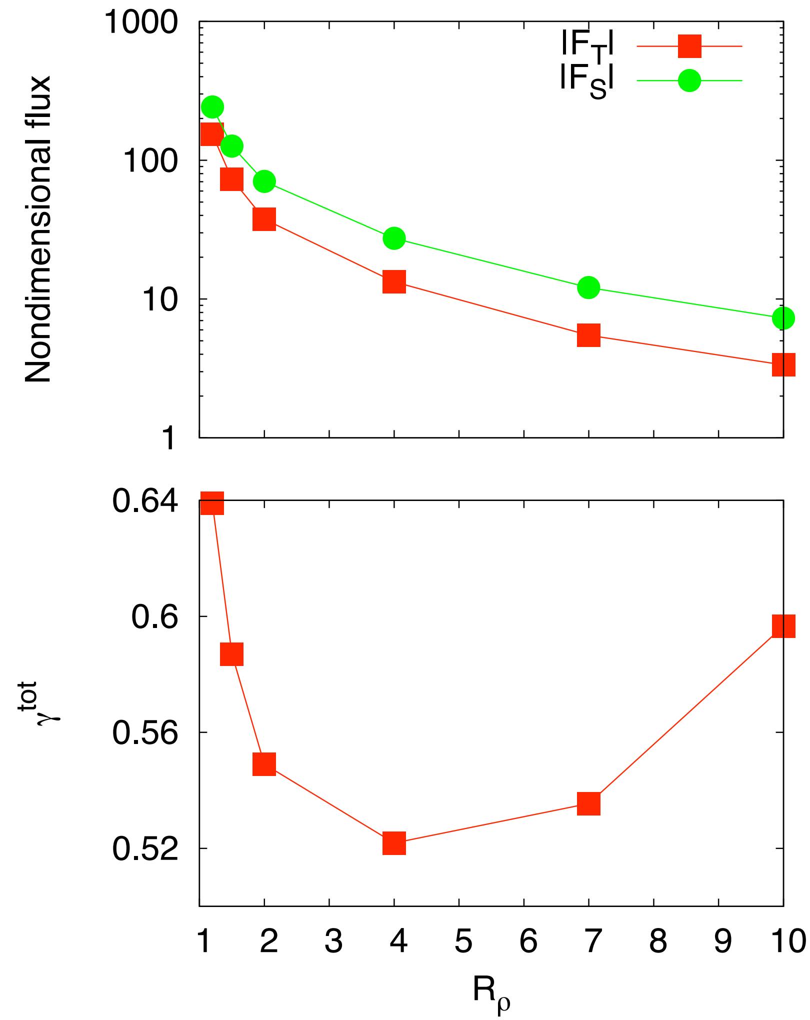


From Traxler et al. 2011

Mixing by fingering convection for salt water

Results:

- Flux laws for salt water can be determined experimentally with this method, and used as parametric laws in global ocean models.
- Typical "turbulent diffusivities" found of order of 100 times larger than microscopic, consistent with actual oceanic measurements



From Traxler et al. 2011

Mixing by fingering convection in astrophysics

- In astrophysical systems, typical parameters Pr and τ are <<1 because thermal diffusion increased by photon transport while other diffusion coefficients are not.

- Planetary interiors: $\mathrm{Pr}, \tau \approx 10^{-3}$
- Stellar interiors: $\mathrm{Pr}, \tau \approx 10^{-6}$

- The stellar parameter regimes is not achievable numerically – scale separation too large. Planetary regime maybe approachable.

Mixing by fingering convection in astrophysics

- We ran a series of numerical experiments for gradually decreasing values of Pr, and τ between 0.1 and 0.03.

Set	1	2	3	4	5	6
Pr	1/3	1/3	1/10	1/10	1/10	1/30
τ	1/3	1/10	1/3	1/10	1/30	1/10

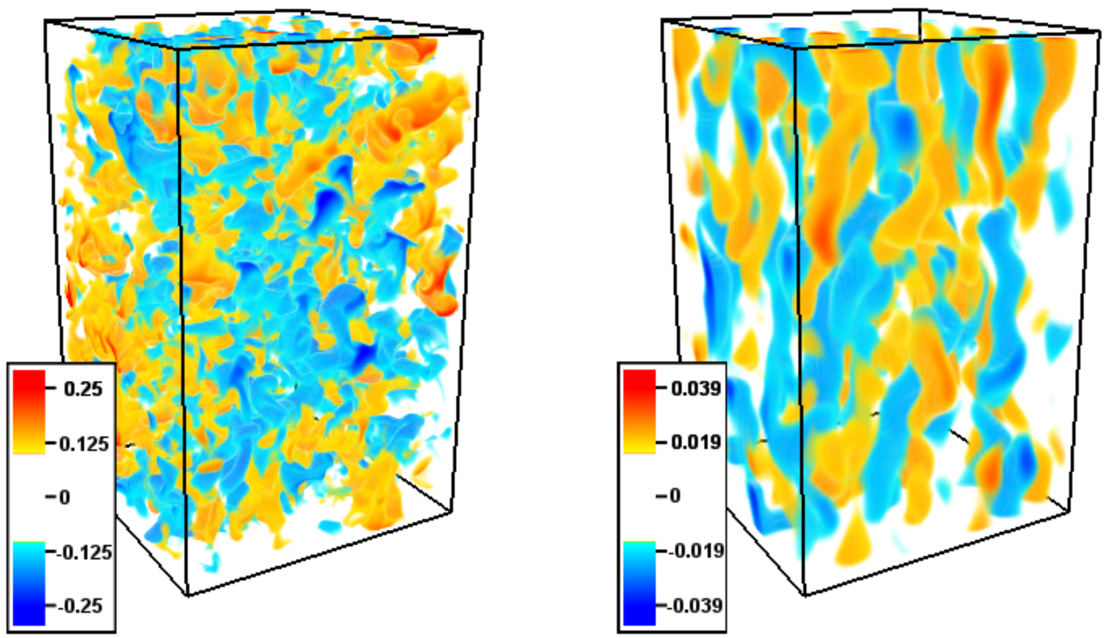
- In each case, density ratio is varied across whole instability range $1 < R_0 < \tau^{-1}$

Mixing by fingering convection in astrophysics

- General structure of fingering convection similar to salt-water case, although details differ…

$\mathrm{Pr} = 0.1$

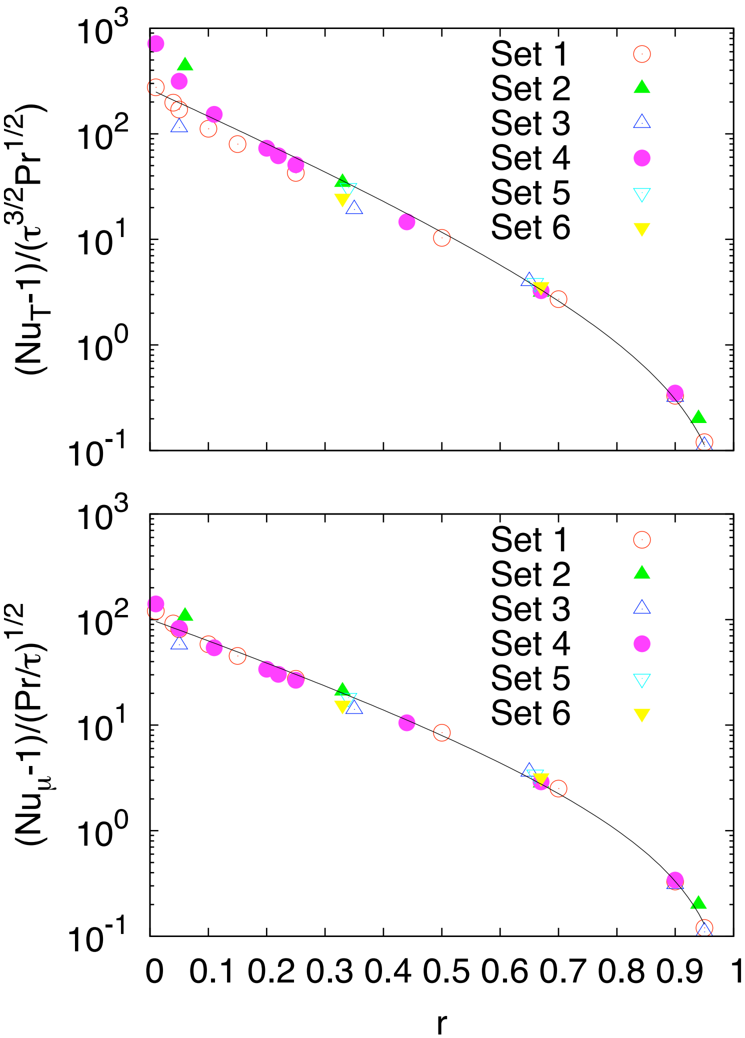
$\tau = 0.1$



$R_0 = 1.45$ $R_0 = 9.1$

Traxler et al. 2011

Mixing by fingering convection in astrophysics



Result: Turbulent heat and compositional transport follow *universal asymptotic law*:

$$\text{If} \quad Pr \ll 1, \quad \tau \ll 1, \quad Pr \sim \tau$$

$$Nu_T - 1 = Pr^{1/2}\, \tau^{3/2} F(r)$$

$$Nu_\mu - 1 = \sqrt{\frac{Pr}{\tau}}\, G(r)$$

$$\text{where } r = \frac{R_0 - 1}{R_c(Pr,\tau) - 1}$$

Traxler et al. 2011

Mixing by fingering convection in astrophysics

Physical implications:

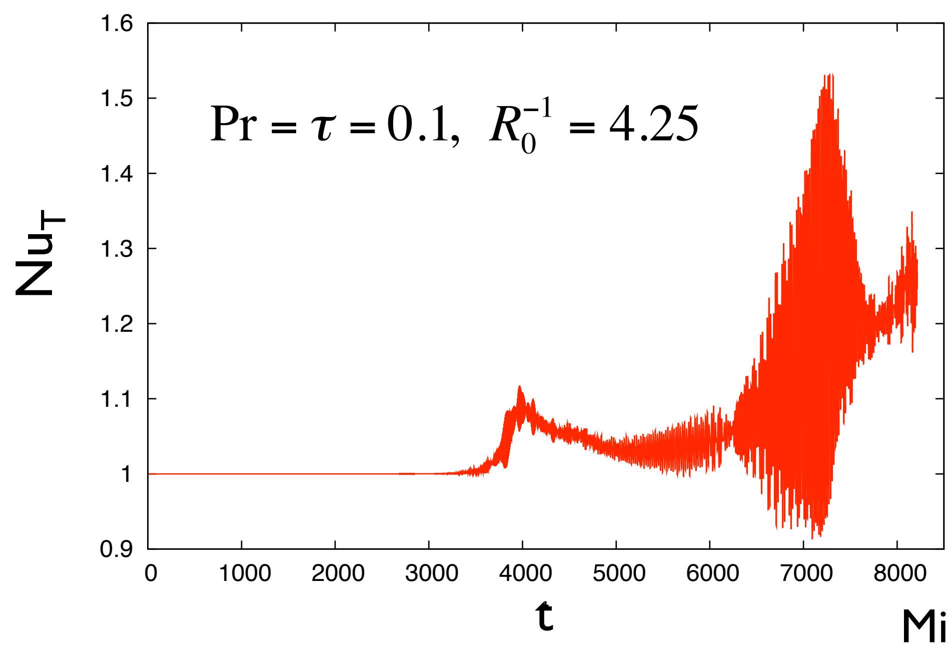
Turbulent diffusivity = (Nu-1) (microscopic diffusivity).

- $\mathrm{Nu}_T - 1 = \mathrm{Pr}^{1/2}\,\tau^{3/2} F(r)$: turbulent heat transport by fingering convection typically negligible in stellar or planetary interiors.
- $\mathrm{Nu}_\mu - 1 = \sqrt{\mathrm{Pr}/\tau}\, G(r)$: turbulent compositional transport by fingering convection up to a few orders of magnitude larger than diffusion in stellar/planetary interiors. **New law can be used "as is" in evolution codes.**

Mixing by oscillatory convection in astrophysics

Preliminary results:

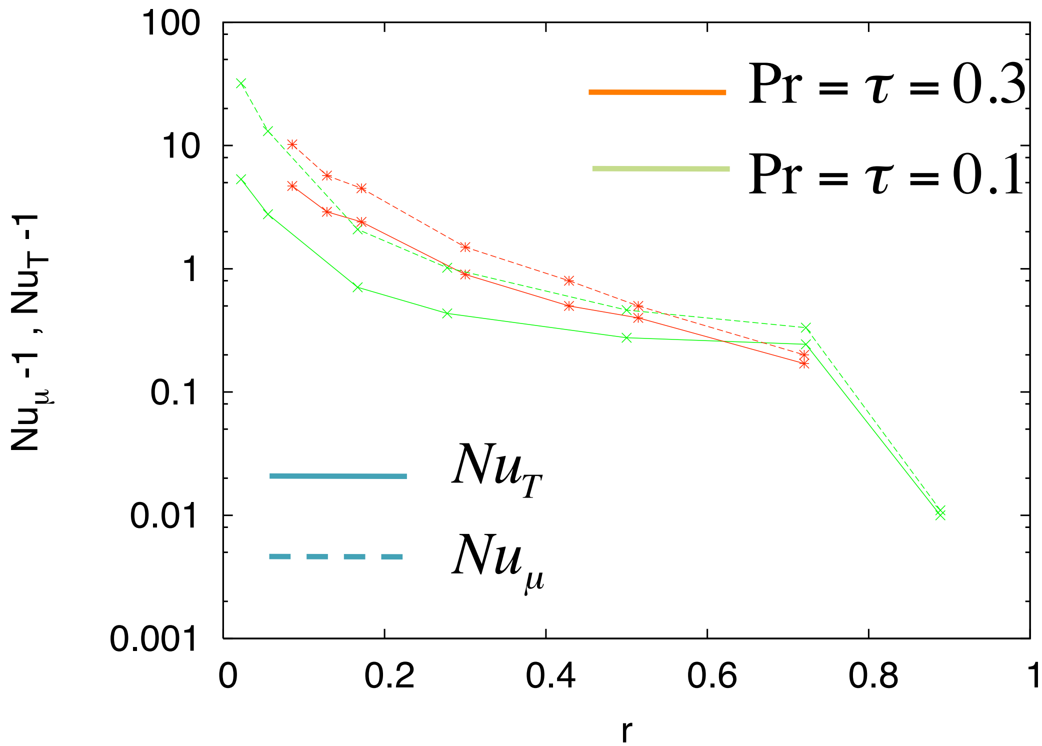
- Similar setup as in the fingering case
- *Caveat*: in this case the Nusselt numbers sometimes do not appear to give statistically steady transport values:



Mirouh et al., in prep.

Mixing by oscillatory convection in astrophysics

Preliminary results: turbulent mixing does not appear to follow same asymptotic law as fingering convection. More runs needed to determine what law it actually is.



$$r = \frac{R_0^{-1} - 1}{R_c^{-1}(\mathrm{Pr},\tau) - 1}$$

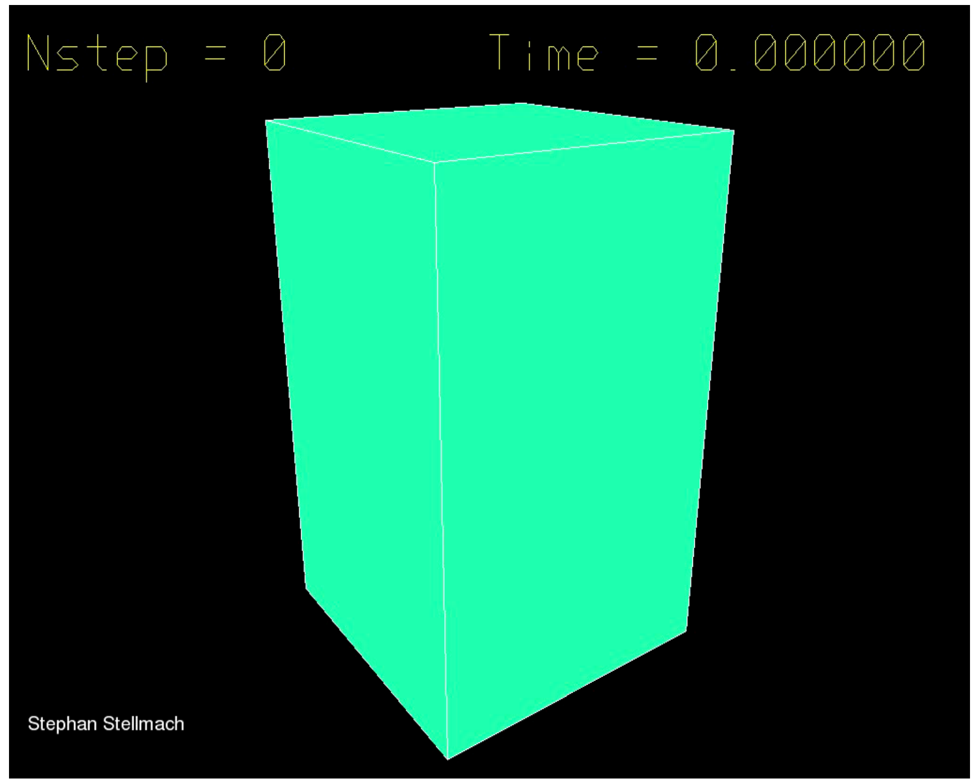
Mirouh et al., in prep.

Outstanding questions

- Transport by small-scale instabilities
- The formation of large-scale structures
- Transport through a staircase

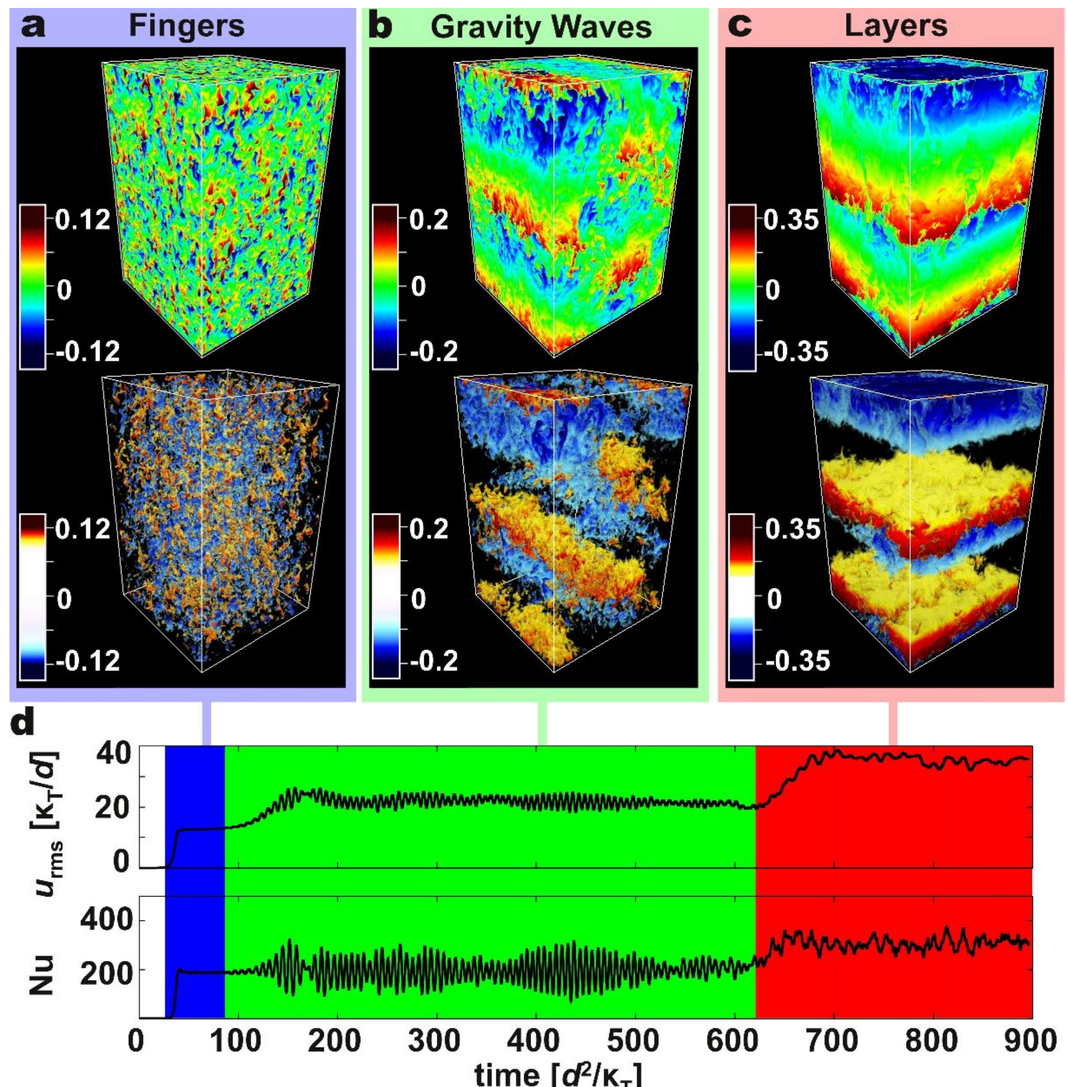
Large-scale dynamics in fingering convection

- All previous runs were done in "small" domain sizes (5x510 FGW).
- Let's look at a larger domain run
  - 25 x 25 x 40 FGW domain size,
  - Pr = 7, $\tau = 1/3$
  - $R_0 = 1.1$ (close to convective overturning instability)



Stellmach et al. 2011

Large-scale dynamics in fingering convection



Stellmach et al. 2011

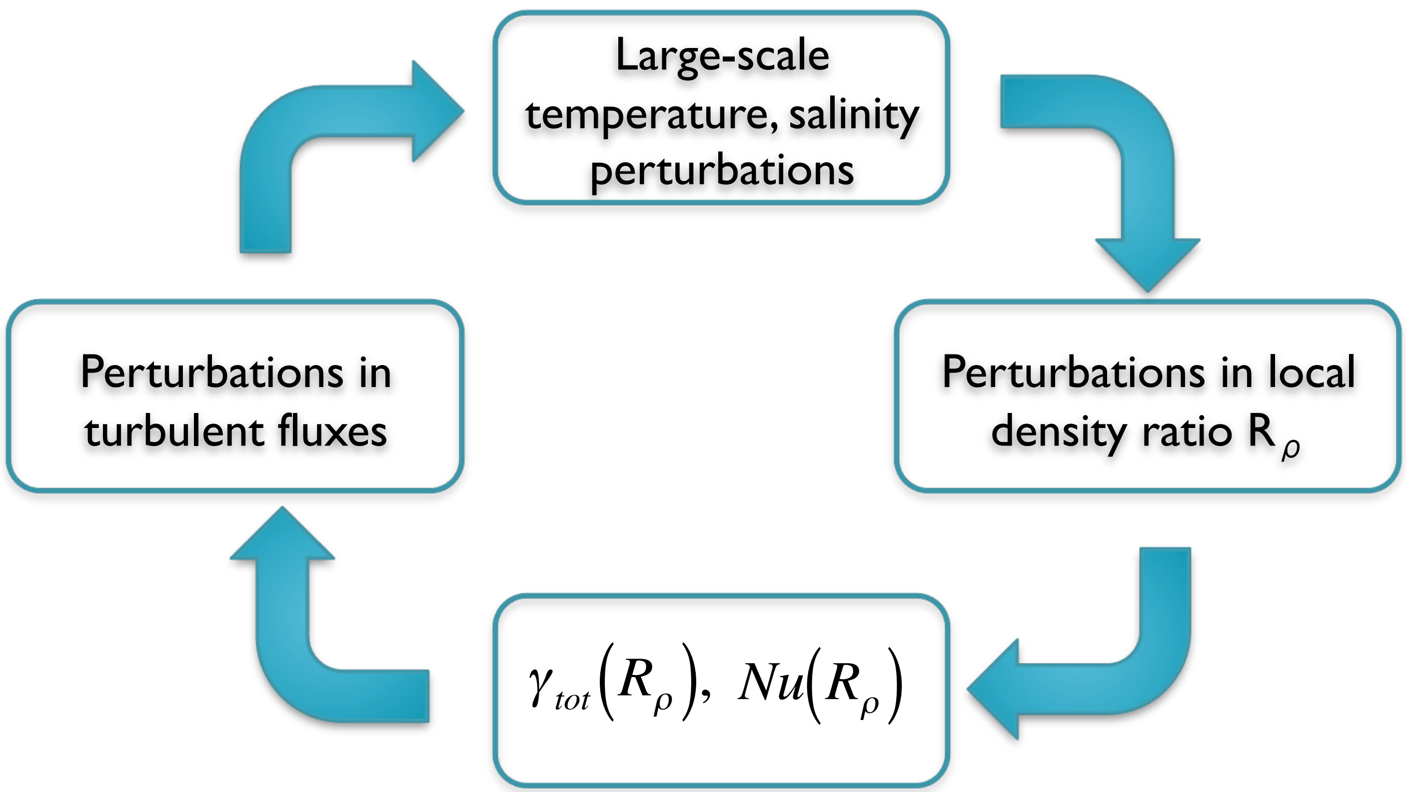
Large-scale dynamics in fingering convection

- Emergence of large-scale dynamics (gravity waves and layers) can be understood using "mean-field" theory
  - Long tradition of this approach for fingering convection: Stern & Turner, 1969; Walsh & Ruddick, 1995; Stern et al. 2001; Radko 2003. …

- Mean-field theory
  - Note that emerging structure scale >> finger scale
  - Spatially average governing equations over small scales
  - Use empirically motivated closure to model turbulent transport by the small-scales
  - Study the resulting evolution of the large-scale fields

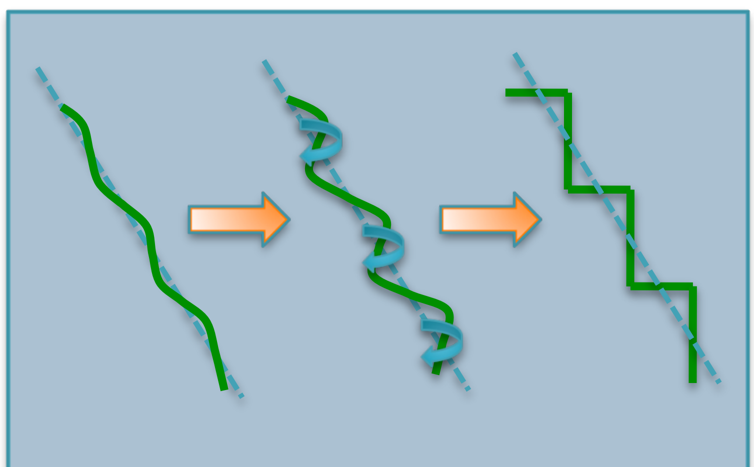
Large-scale dynamics in fingering convection

- Spontaneous formation of large-scale structures induced by positive feedback between large-scale temperature/salinity perturbation and induced fluxes.



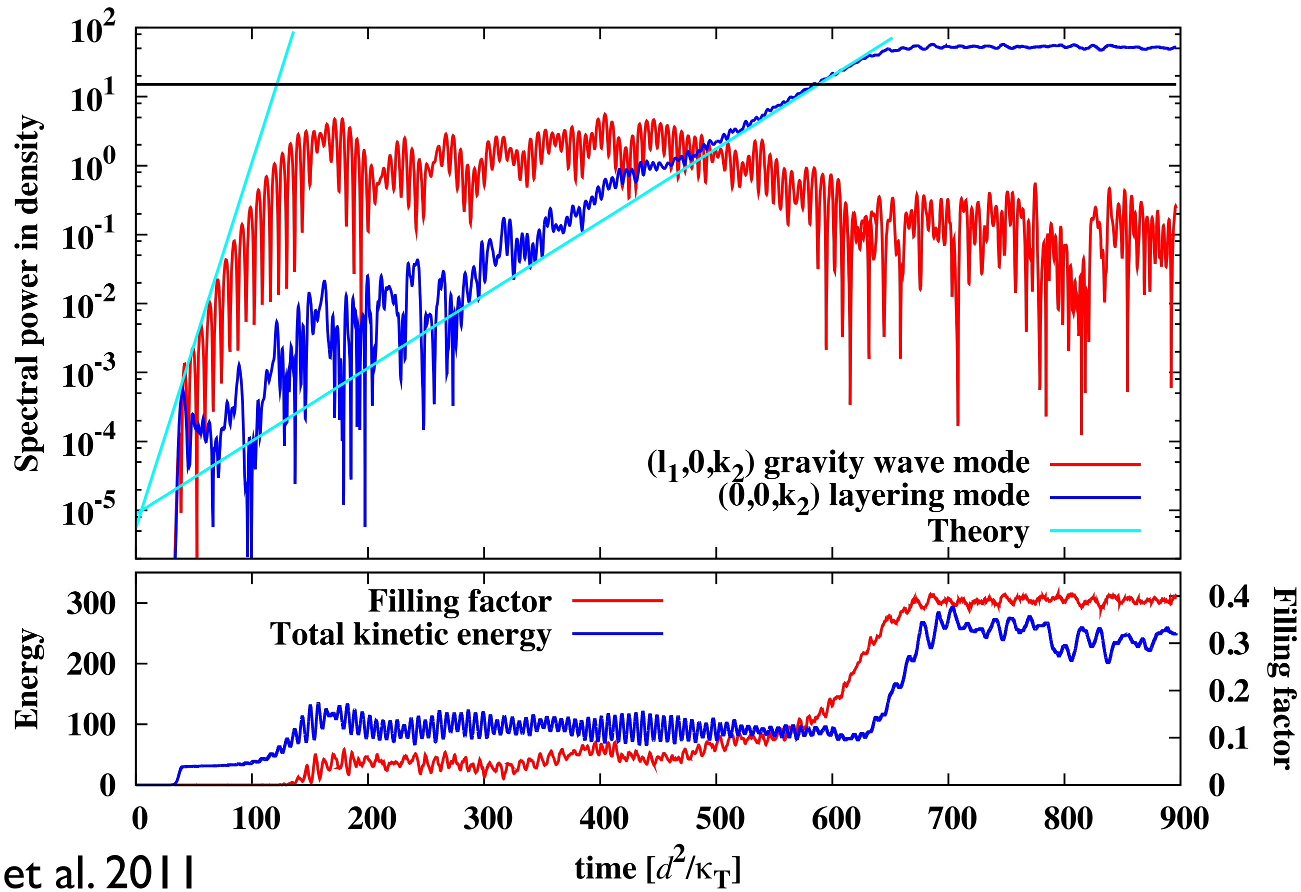
Large-scale dynamics in fingering convection

- Theory predicts:
  - Large-scale gravity wave excitation if $Nu(R_\rho)$ is large enough
  - Large-scale layering modes if $\gamma_{tot}(R_\rho)$ is a decreasing function
  - Mode growth rates depend on $Nu(R_\rho)$ and $\gamma_{tot}(R_\rho)$
- Layering mode overturns into a staircase when amplitude is large enough



Large-scale dynamics in fingering convection

- Comparison with large-domain numerical simulations reveals very good agreement with theory …

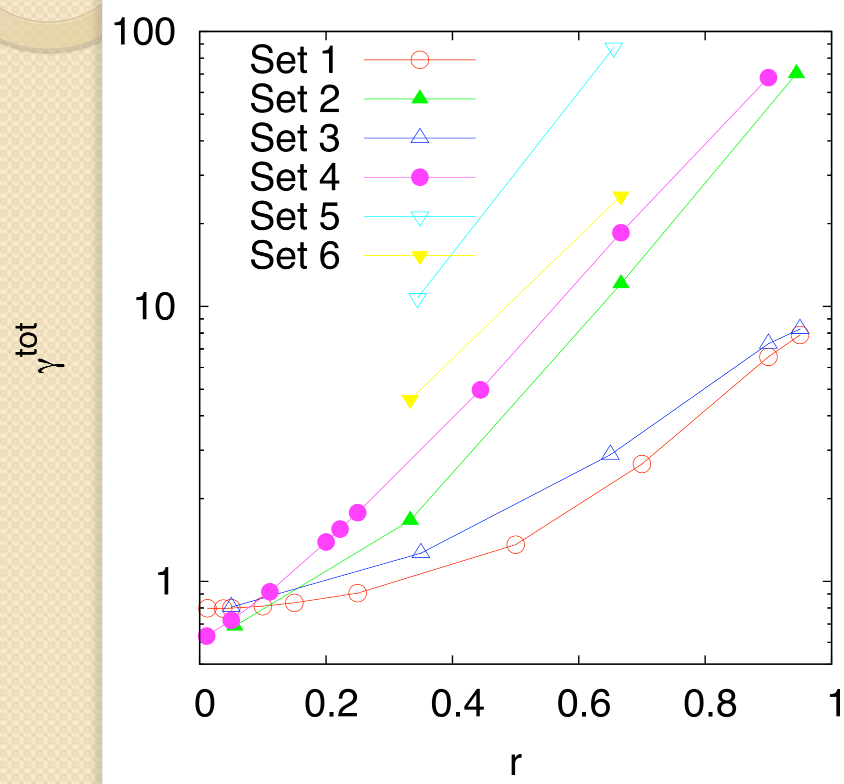


Stellmach et al. 2011

Large-scale dynamics in fingering convection

- **In short:**
  - We can now predict, for any parameter regime, whether large-scale gravity waves and/or layers will form, and at what rate they grow
  - What is the initial spacing of the staircase
  - The only requirement is to measure small-scale transport laws from small-domain simulations…
- **Note that:**
  - Mean-field theory can equally well be applied to the oscillatory regime. Predicts layer formation also when γ^{-1} is a decreasing function of R^{-1}

Large-scale dynamics in fingering convection in the astrophysical regime



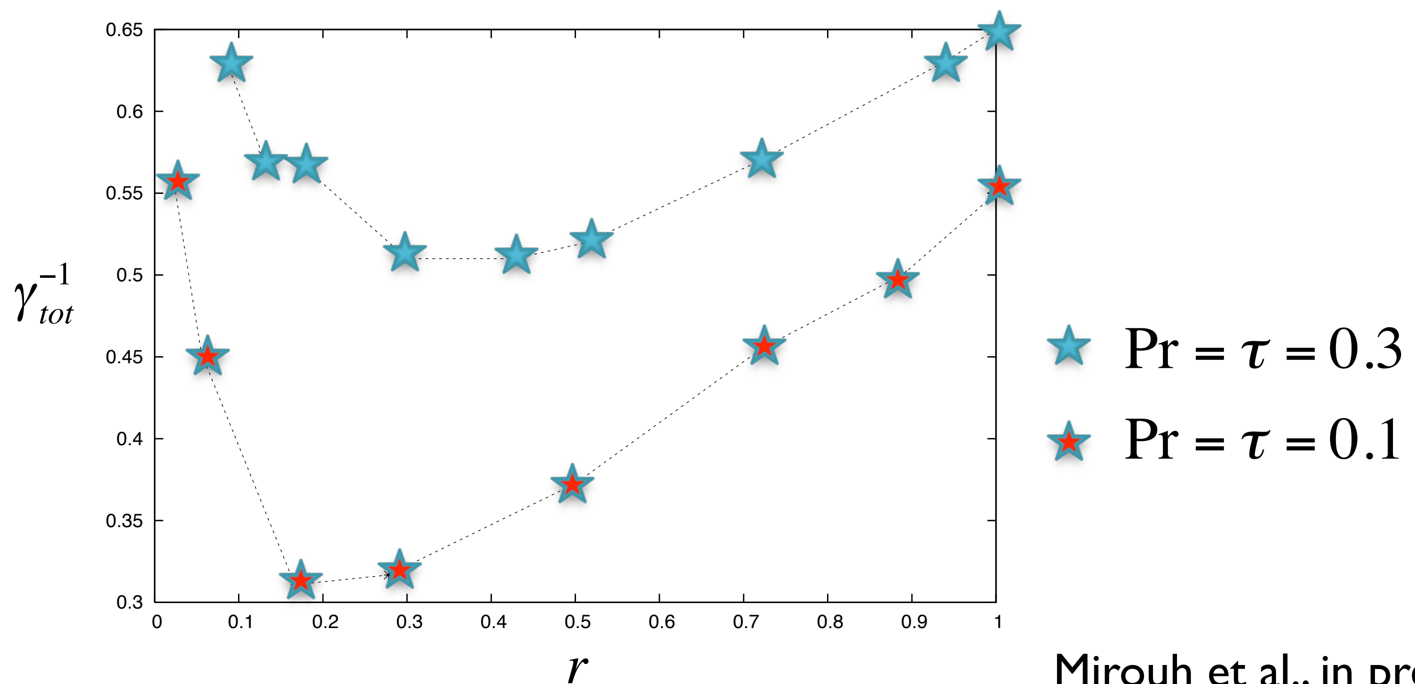
- For low Pr, we find that:
  - Nu is very small
  - γ_{tot} "never" decreases with R_ρ
- Implications for low Pr fingering systems:
  - **No spontaneous staircase formation through γ – instability mechanism**
  - **No large-scale gravity waves**
  - **Transport properties dominated by small-scale dynamics**

Traxler et al. 2011

Large-scale dynamics in oscillatory convection in the astrophysical regime

Results:

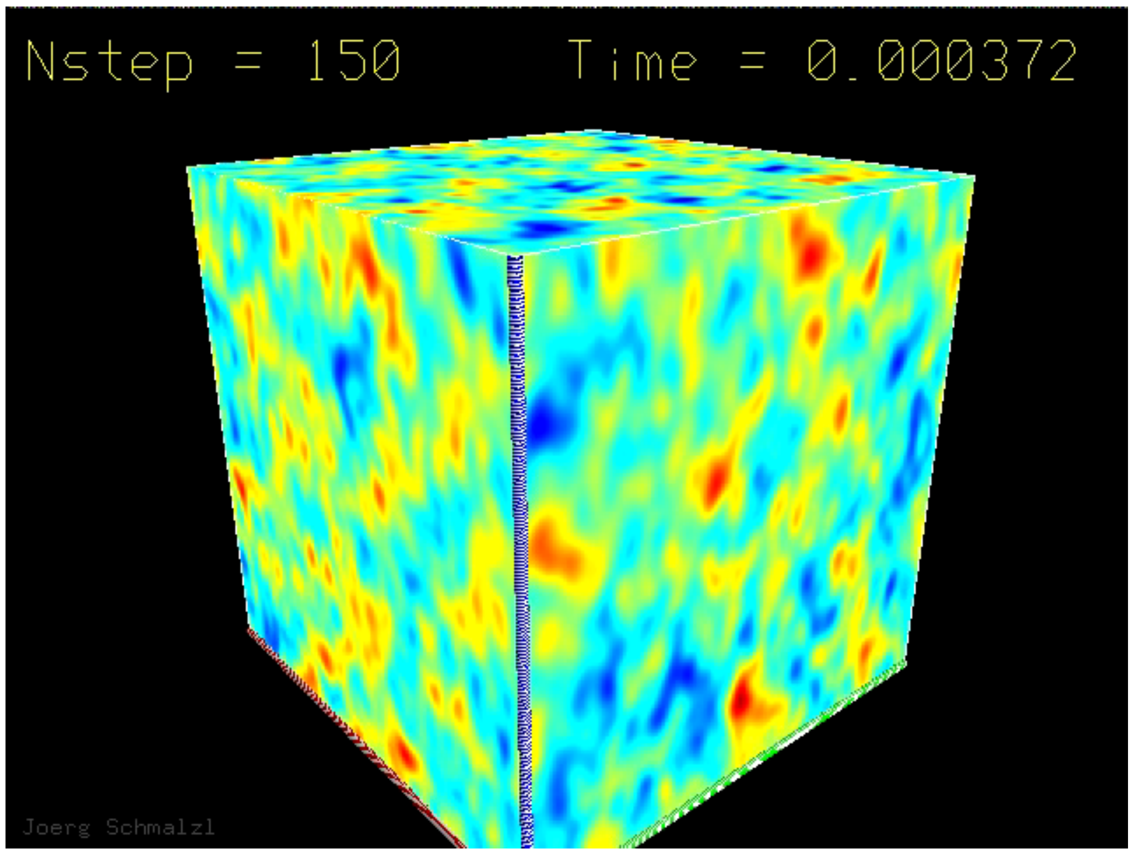
- In this case, for low Pr, γ_{tot} does decrease for low R_ρ so **staircase formation is expected in weakly stratified systems**



Mirouh et al., in prep.

Large-scale dynamics in oscillatory convection in the astrophysical regime

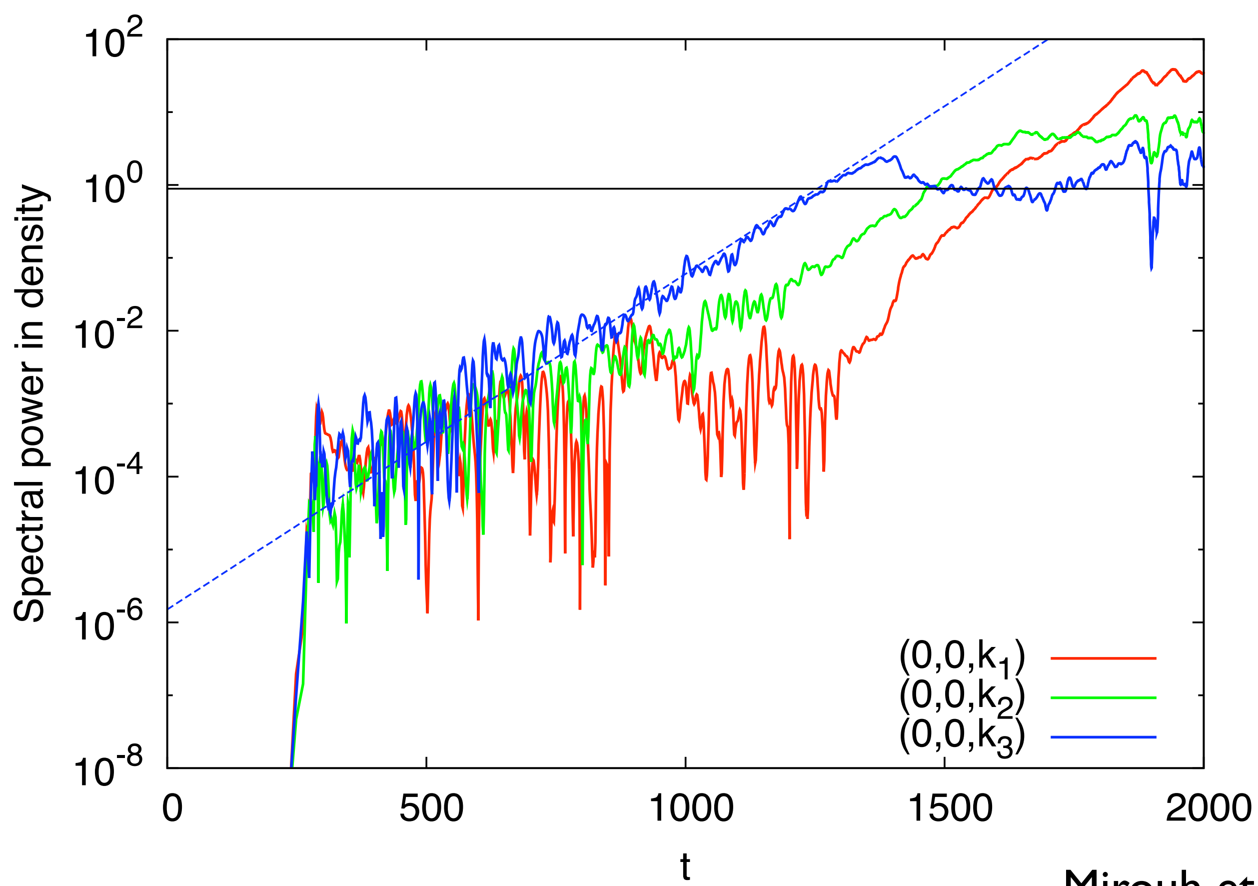
- Simulation for $\mathrm{Pr} = \tau = 0.3, R_0^{-1} = 1.2$



Mirouh et al., in prep.

Large-scale dynamics in oscillatory convection in the astrophysical regime

- Layer formation in this regime is also explained by Radko's γ-instability:



Mirouh et al., in prep.

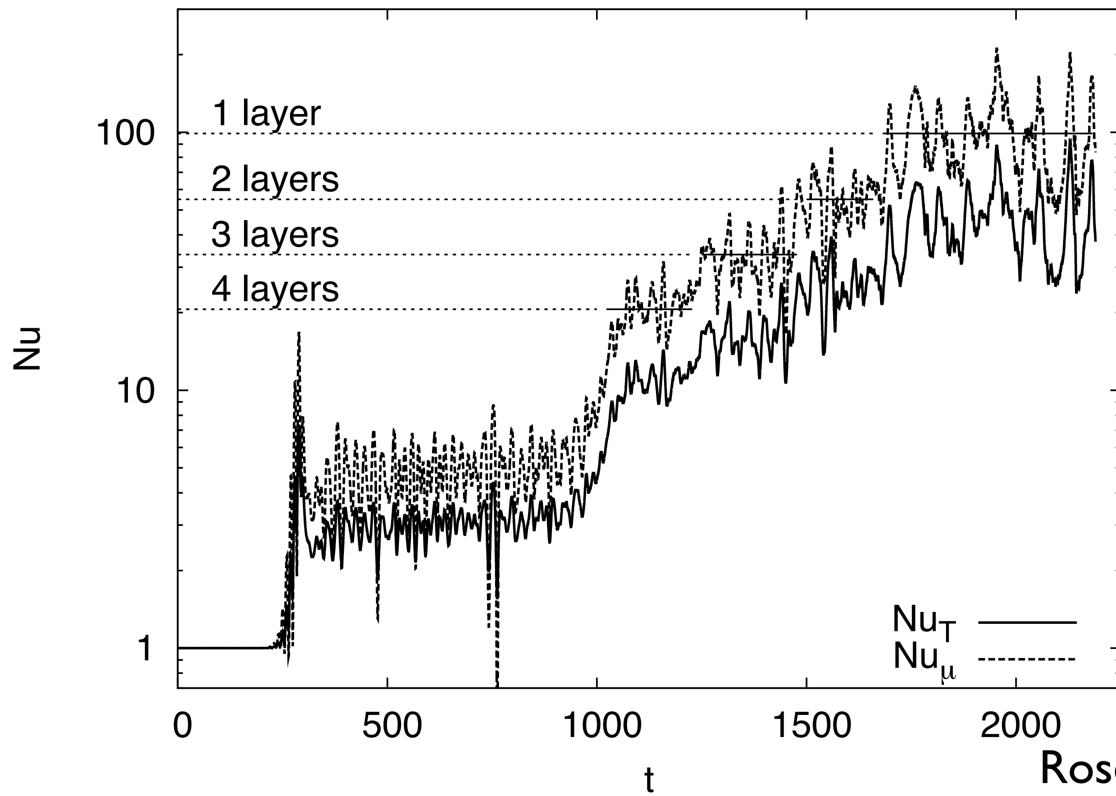
Outstanding questions

- Transport by small-scale instabilities
- The formation of large-scale structures
- Transport through a staircase

Staircase transport in oscillatory convection in the astrophysical regime

Results:

- Staircase formation and each subsequent merger increases effective Nu.



Rosenblum et al. 2011.

Staircase transport in oscillatory convection in the astrophysical regime

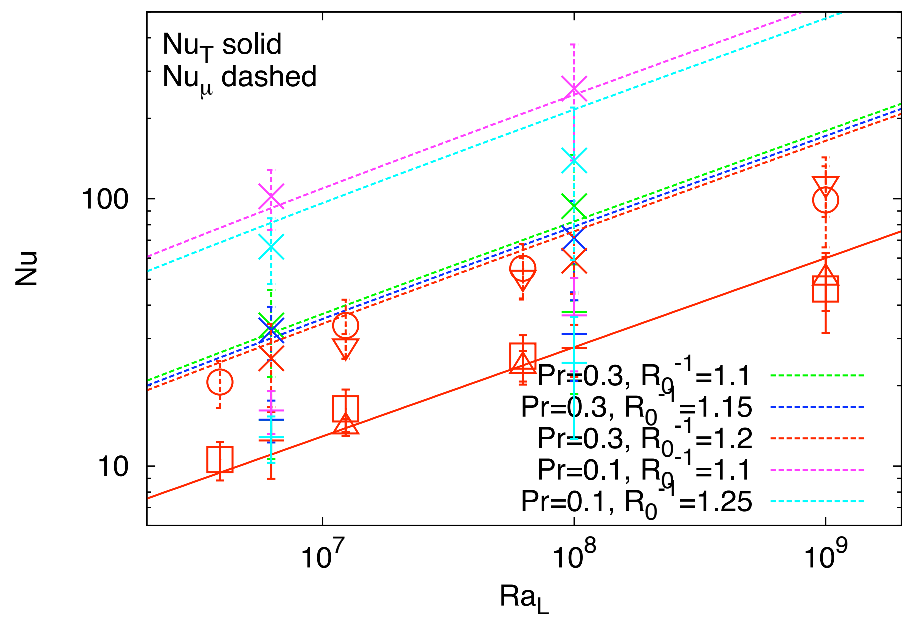
Results:

- Transport properties in the layered convection case seems to be “well” explained with Rayleigh-Benard scaling laws based on layer height h

$$Ra_L = \frac{\alpha g |T_{0z}| h^4}{\nu \kappa_T}$$

$$Nu_T = 0.06 Ra_L^{1/3}$$

$$Nu_\mu = 1 + \frac{R_0}{\tau}(Nu_T - 1)$$

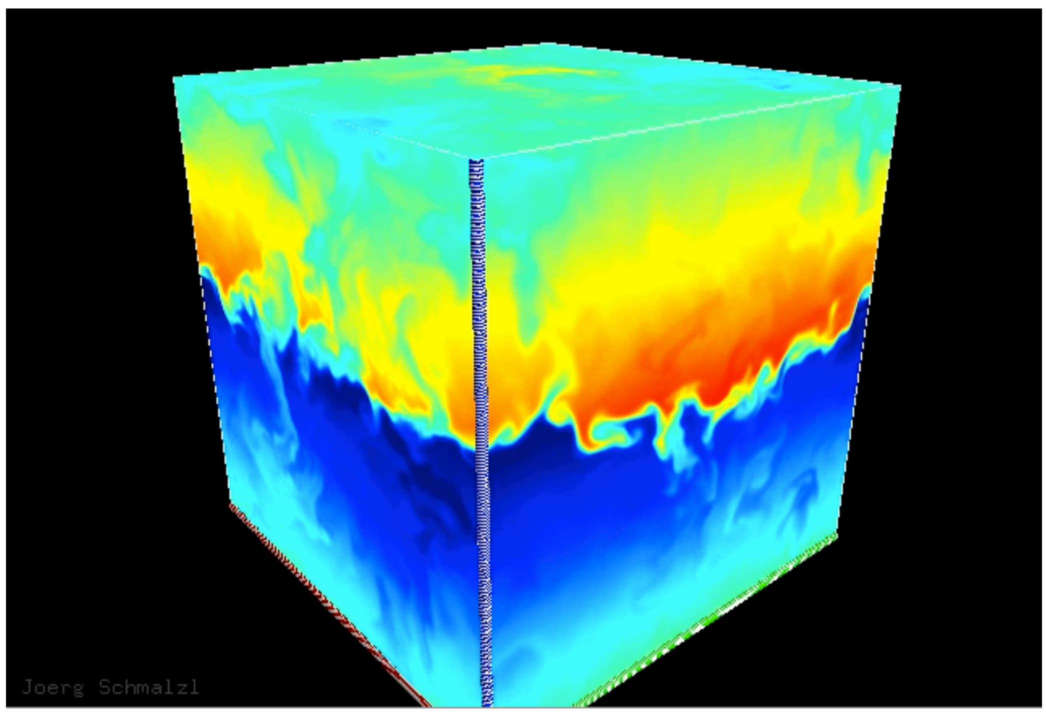


Rosenblum et al 2011, Mirouh et al., in prep.

Staircase transport in oscillatory convection in the astrophysical regime

However, what determines the ultimate layer height?

- In simulations, final layer is always as tall as box
- In real objects? To be determined….



Conclusions/prospects

Conclusions/prospects

- We are well on the way to having a comprehensive theory of double-diffusive convection in astrophysics, in both fingering and oscillatory regimes

- **Fingering regime:**
  - Asymptotic transport laws have been determined
  - Layer formation and gravity wave formation unlikely
  - **Implications for astrophysics (examples):**
    - Fingering convection unlikely to be sufficient to explain peculiar abundances of AGB stars (Denissenkov, 2011, Traxler et al. 2011)
    - Fingering convection plays an important role in metallicity dilution in planet-host stars after impact by a planet, and possible role on Li depletion (Vauclair, 2004; Garaud, 2011)

Conclusions/prospects

- We are well on the way to having a comprehensive theory of double-diffusive convection in astrophysics, in both fingering and oscillatory regimes

- **Oscillatory regime**
  - Asymptotic transport laws remains TBD
  - Gravity wave excitation unlikely
  - Layer formation likely
  - Potential transport law through staircase established, but equilibrium layer height remains TBD.
  - **Implications for astrophysics (examples):**
    - Possible role of layered convection in explaining the diversity of gas giant planet heat fluxes / radii. (Chabrier & Baraffe, 2007)

Traxler, Stellmach, Garaud, Radko & Brummell 2011 (JFM)

The dynamics of fingering convection I: Small-scale fluxes and large-scale instabilities

Stellmach, Traxler, Garaud, Brummell & Radko 2011 (JFM)

The dynamics of fingering convection II: The formation of thermohaline staircases

Traxler, Garaud & Stellmach, 2011 (ApJL)

Turbulent transport by fingering convection in astrophysics

Rosenblum, Garaud, Traxler & Stellmach 2011 (ApJ)

Layer formation and evolution in semi-convection

Extras

Large-scale dynamics in fingering convection

- Averaged equations:

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \overline{u}}{\partial t} + \nabla \cdot R\right) = -\nabla \overline{p} + \left(\overline{T} - \overline{S}\right)\mathbf{e}_z + \nabla^2 \overline{u}$$

$$\frac{\partial \overline{T}}{\partial t} + \nabla \cdot F_T + \overline{w} = \nabla^2 \overline{T}$$

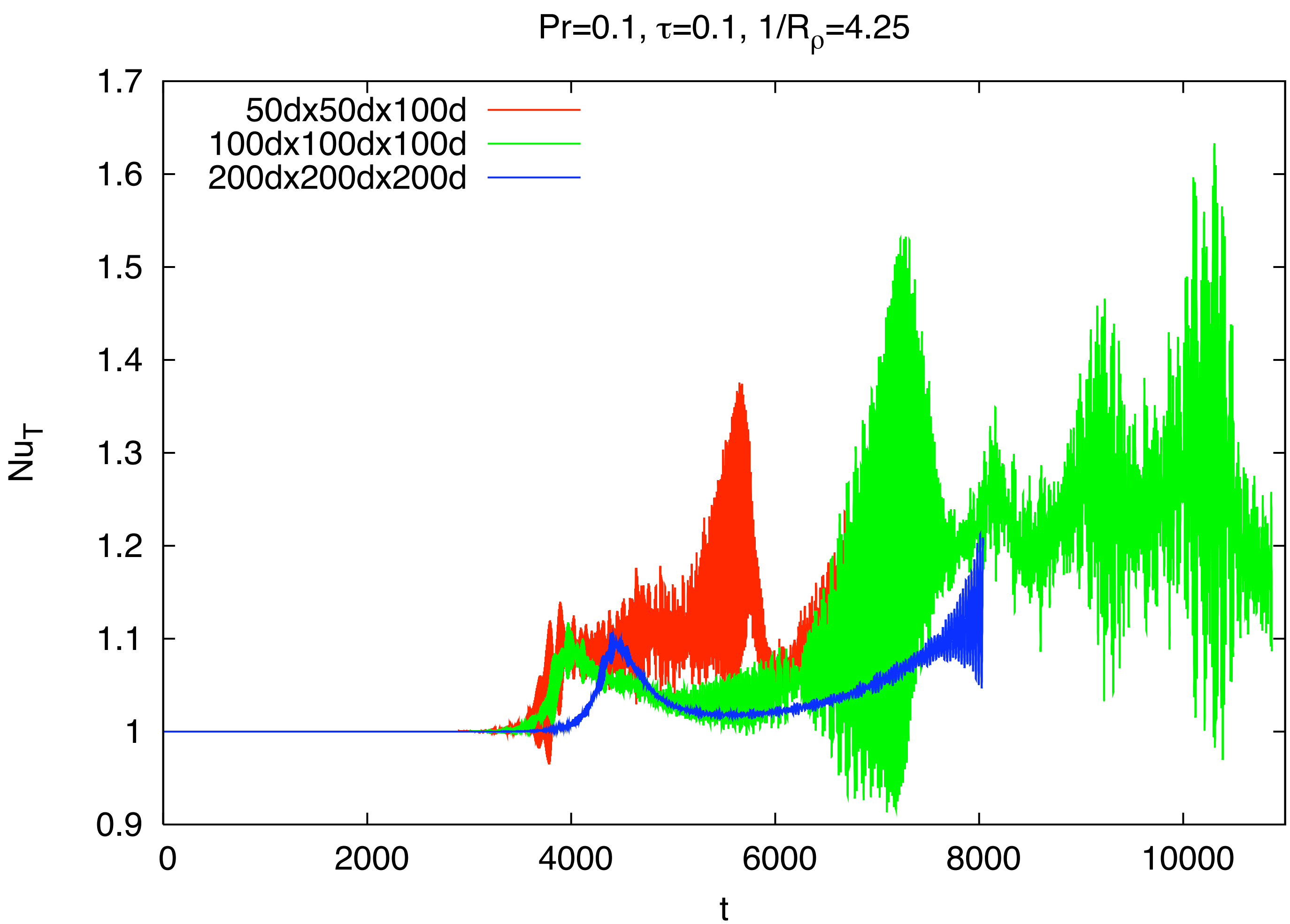
$$\frac{\partial \overline{S}}{\partial t} + \nabla \cdot F_S + \frac{\overline{w}}{R_0} = \tau \nabla^2 \overline{S}$$

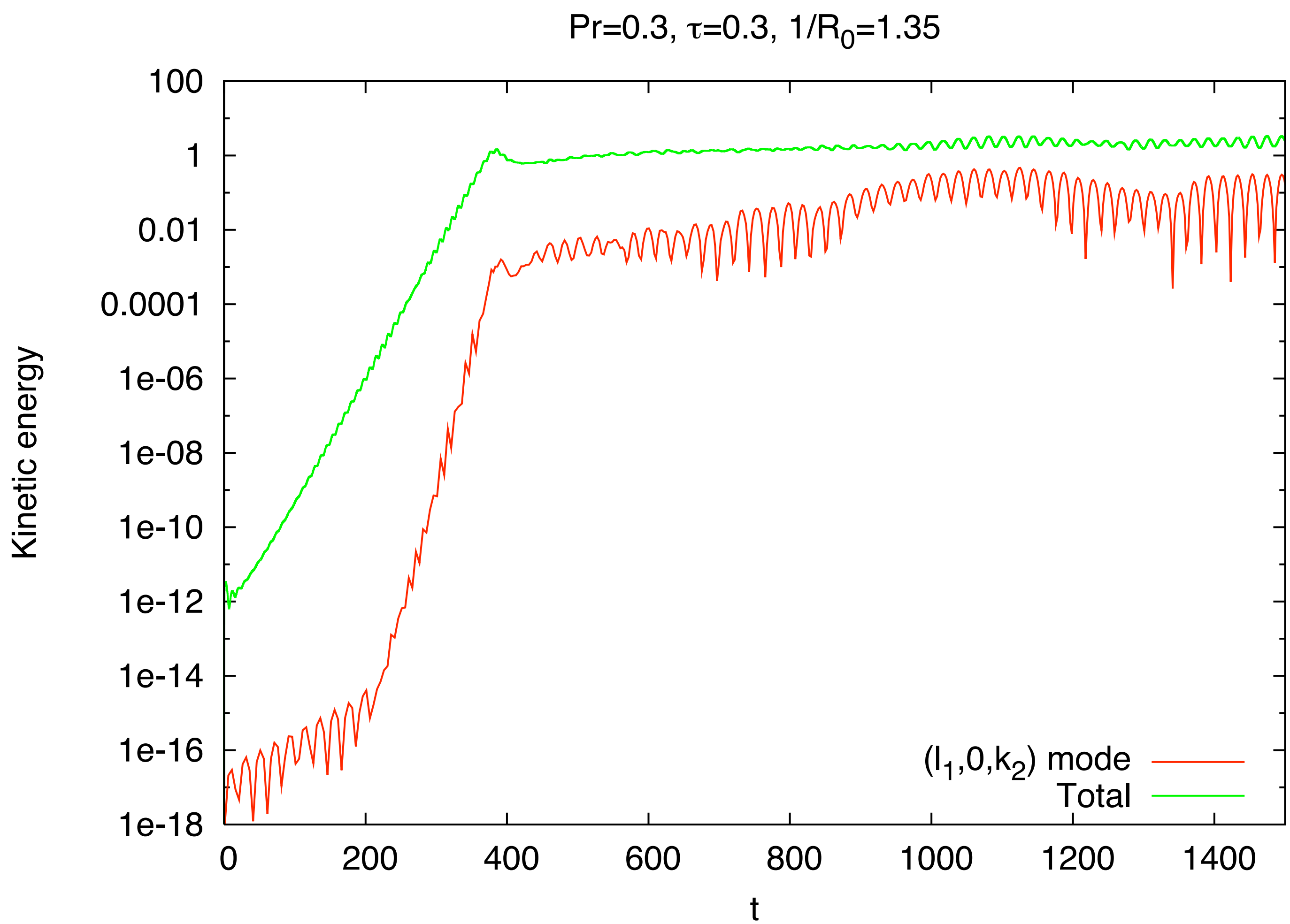
$$\nabla \cdot \overline{u} = 0$$

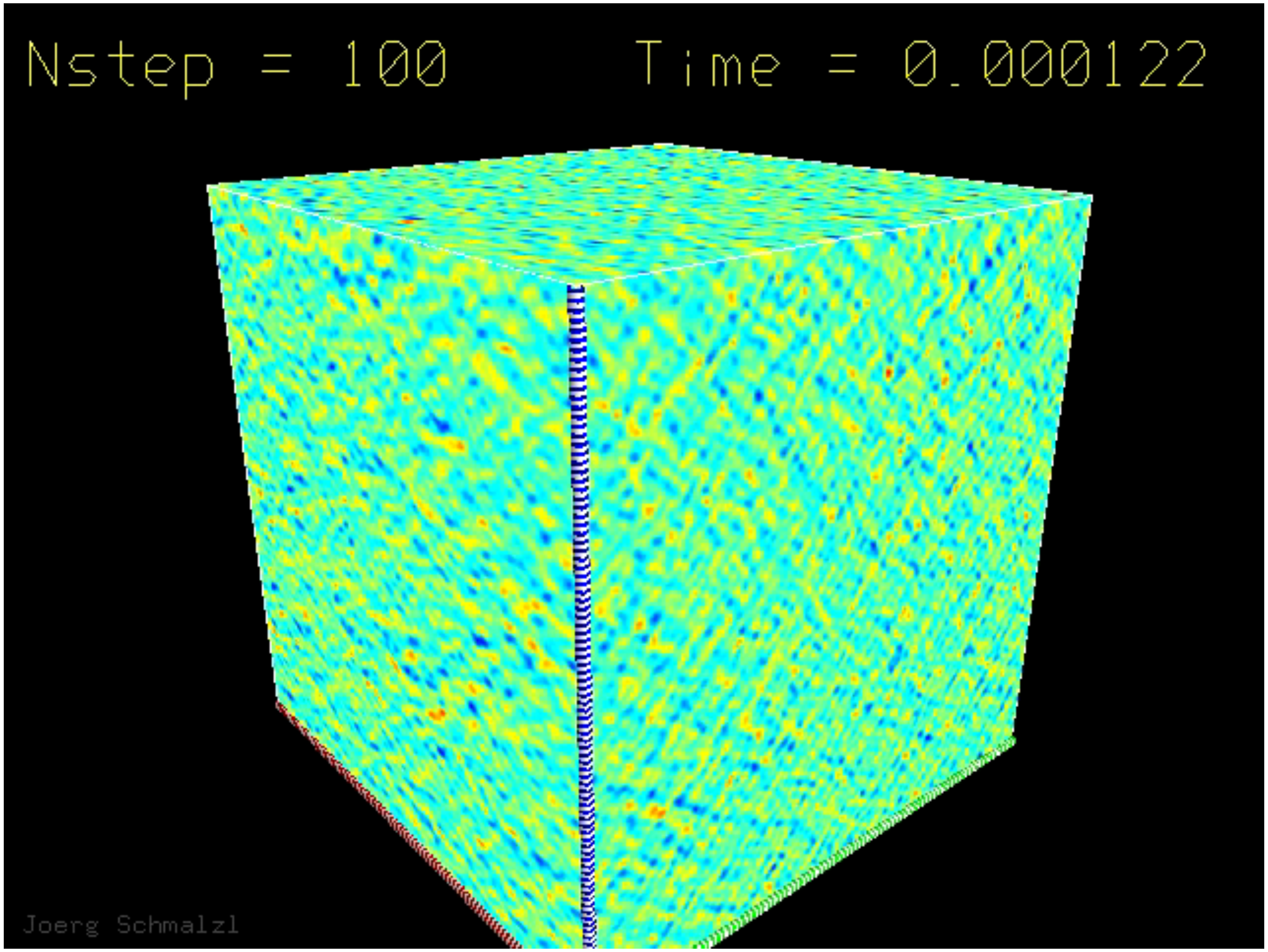
- Flux model:
  - Neglect Reynolds stresses
  - Salinity, heat fluxes only have vertical component
  - Non-dimensional fluxes only depend on local

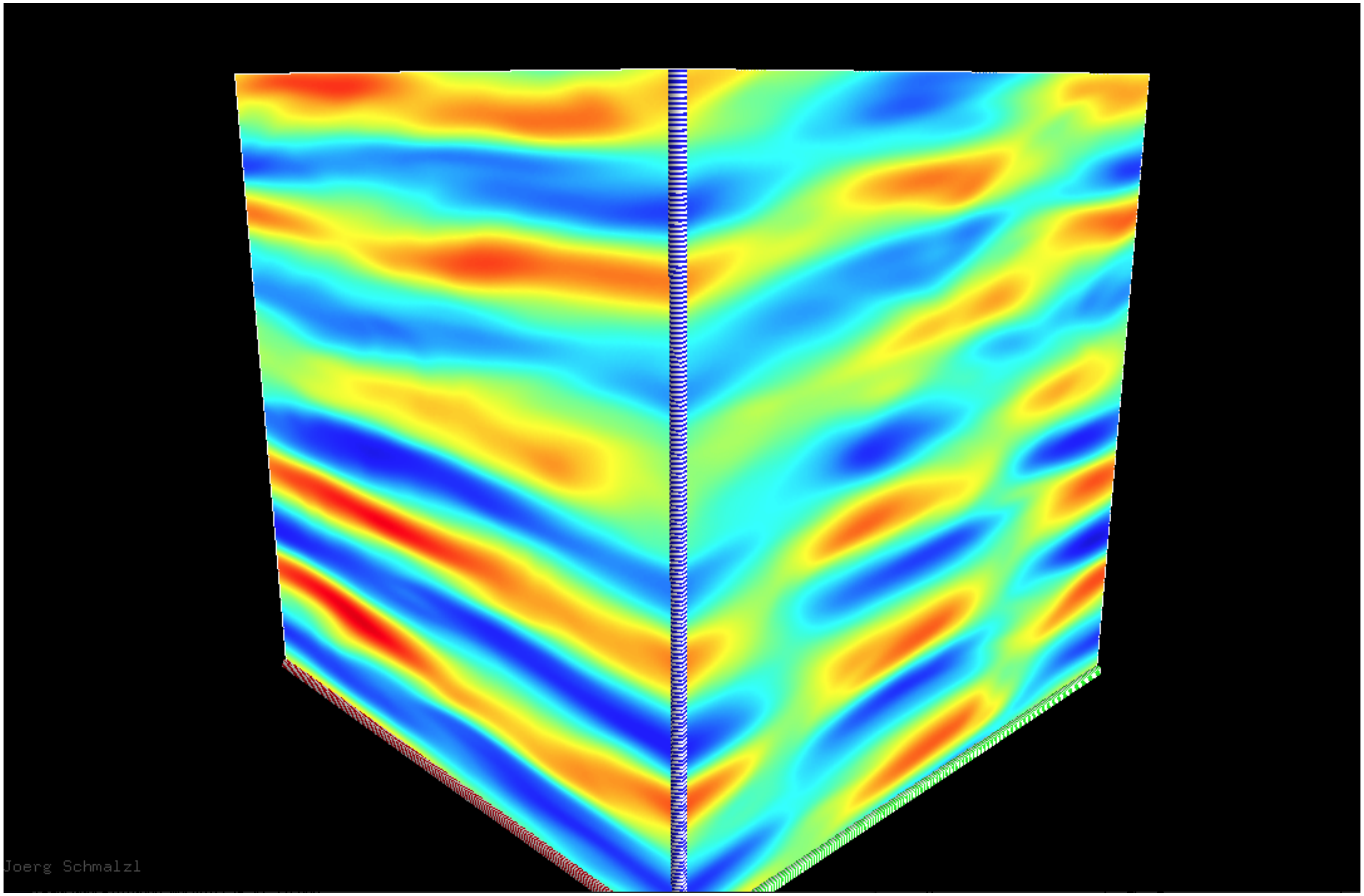
$$R_\rho = \frac{T_{0z} + \overline{T}_z}{S_{0z} + \overline{S}_z}$$

- Non-dimensional fluxes:
  - Measured with Nu and γ_{tot}
  - Empirically known from small box simulations!


Pr=0.1, τ=0.1, 1/Rρ=4.25
50dx50dx100d
100dx100dx100d
200dx200dx200d
NuT
t
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0
2000
4000
6000
8000
10000


Pr=0.3, τ=0.3, 1/R0=1.35
Kinetic energy
100
1
0.01
0.0001
1e-06
1e-08
1e-10
1e-12
1e-14
1e-16
1e-18
0
200
400
600
800
1000
1200
1400
t
(l1,0,k2) mode
Total


Nstep = 100
Time = 0.000122
Joerg Schmalzl


Joerg Schmalzl