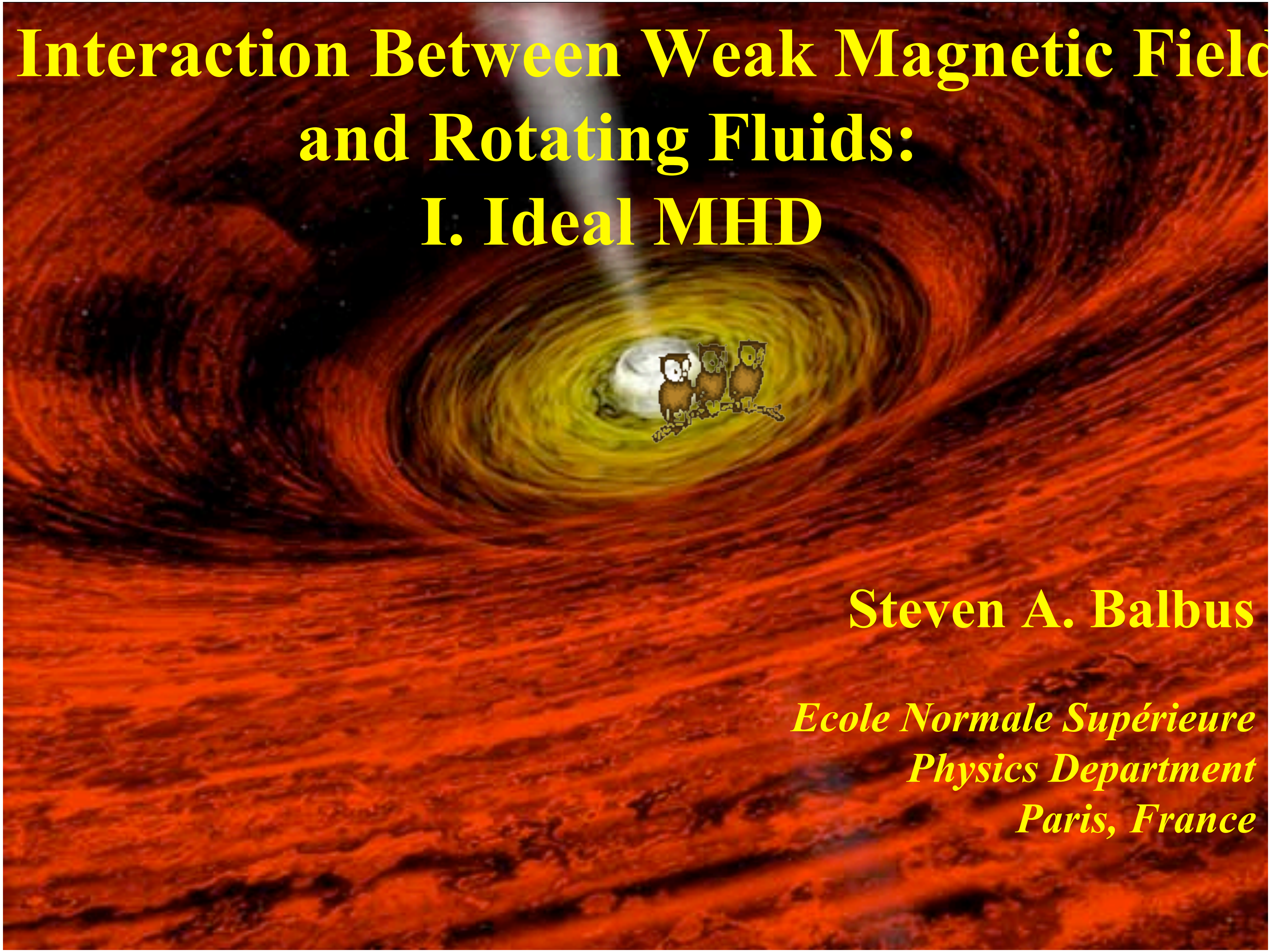
Interaction Between Weak Magnetic Field and Rotating Fluids: I. Ideal MHD



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FORMATION and EVOLUTION OF PROTOSTELLAR DISKS

Problem is central to understanding star and planet formation.

Initial collapse of a rotating molecular cloud is likely to be planar, not spherical, with a condensed pressurized core and an extended rotating reservoir of angular momentum. Identify with protostar and protostellar disk.

Early infall stages will be strongly self-gravitating. Deep spiral structure in disk effective at angular momentum transport. Disk turbulence by infall will *not* be effective, for reasons we shall see.

FUNDAMENTALS OF ACCRETION DISKS

- Modern accretion disk theory began in the 1960's from the attempts of Lynden-Bell, Rees, Thorne and others to establish observational signatures from hot gas in orbit around black holes.

- Applications of this formalism to the solar nebula (and protostellar disks more generally), pioneered by Lin and Papaloizou, began in earnest around 1980.

- A quarter of a century later, much remains elusive.

PROBLEM #1: ANGULAR MOMENTUM

• The classical problem for accretion sources is that astrophysical gases all have some angular momentum. This creates a $1/r^2$ repulsive potential when the gas tries to coalesce, as opposed to a $1/r$ attractive gravitational potential.

• All the matter needs to come to the center, all the angular momentum needs to be exported to infinity. It is not obvious how to do this.

• Spiral galaxies are, in fact, the contortions of a self-gravitating system trying to do just that *(Lynden-Bell & Kalnajs 1972).*

PROBLEM #1: ANGULAR MOMENTUM

• What if there is no self-gravity? Some sort of angular momentum transport must be taking place. It is often called “anomalous viscosity”.

But do orbiting planets have “anomalous momentum?”

Does the sun have “anomalous conduction” in its outer layers? What is anomalous, anyway…?

One thing about the viscosity: It's certainly not viscosity.

Viscous time scale t is R^2/ν. Molasses in January has a viscosity of about 1000 $cm^2\ s^{-1}$, so t = 10^{18} seconds (3×10^{10} years) for a disk with R ~ ¼ the diameter of the sun. Viscous front velocity is ν/R, about 1 cm per year ~ continental drift velocity!

Viscosity can't even slow down a tea cup: R^2/ν ~ 1600 s ~ 27 mn.

How to account for enhanced momentum transport in disks?

I hate being ‘accounted for’, as if I were some incalculable quantity in an astronomical equation.

--D.L. Sayers, The Documents in the Case, 1929

The chief stumbling block at this point is the friction in the disk. We do not know whether the friction is generated by turbulence in the spiraling gas, by magnetic fields embedded in the spiraling gas, or by a combination of turbulence and magnetic fields.

--K.S. Thorne, Scientific American, 1974

No piece of physics is more poorly understood in the theory of disk accretion than the nature of the viscosity.

--S. Shapiro & S. Teukolsky, Physics of Compact Objects, 1983

Graceful surrender…

The presence of magnetic fields is hardly in doubt (every star has a magnetic field), but the presence of turbulence is more questionable. It is known that medium in nonuniform rotation (Keplerian rotation in our case) in which the angular momentum increases outward is stable against small perturbations. The experiments of Taylor…have, however shown that for large Reynolds number turbulence due to nonlinear effects occurs, for any distribution of angular momentum.

-- Zel'dovich, Ruzmaikin, and Sokoloff, Magnetic Fields in Astrophysics, 1983

Saving face with turbulence:

1) In a turbulent flow, the transport of (angular) momentum is enhanced by orders of magnitude compared with "molecular" transport.

2) In the laboratory, shear flow is generally turbulent when the viscosity ν is small: $Re=UL/\nu >> 1$.

3) But disks also have a large Re!

4) Therefore disks are turbulent, and our problem is solved.

WHAT, EXACTLY, DOES TURBULENCE DO?

Angular momentum flux: $\rho R v_R v_\phi$. But
$v_R = v_d + u_R$, with $<u_R>=0$, and
$v_\phi = v_K + u_\phi$, with $<u_\phi>=0$. The velocity fields consist of a mean part plus a fluctuating part.

The density fluctuations are less important: pretend we are Incompressible. Then:

$$< \rho v_R v_\phi > = \rho v_K v_d + \rho<u_R u_\phi> + \rho v_K<u_R>+ \rho v_d<u_\phi>.$$

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=0   =0

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Direct angular momentum transport advected with flow.

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Turbulent angular momentum transport through the flow.

CONSERVED ANGULAR MOMENTUM FLUX

$R^2 < \rho\, v_R\, v_\phi > = \rho\, R^2\, v_K\, v_d + \rho\, R^2 <u_R u_\phi>=$ Cnst.

The constant is nearly zero, since we may extend this result to small R, and both $v_K\, v_d$ and $<u_R u_\phi>$ grow no faster than $1/R$ (free fall), and very likely much less so.

Hence,

$$v_d = - <u_R u_\phi> / v_K$$

If the u-fluctuations are highly subsonic, and $v_K >> c_S$, then

$$v_d << u << v_K$$

which establishes the asymptotic ordering scheme.

RECAP

1) Turbulence enhances transport in a disk, not because it makes fluctuations, but only if it makes correlated fluctuations, $<u_R\ u_\phi>$. Whether this correlation is present or not is an interesting question that we will address.

2) Turbulence dissipates energy, but not angular momentum. Angular momentum conservation relates v_d to $<u_R\ u_\phi>$ and v_K ; v_d is very small compared to u and v_K.

3) Turbulent transport makes sense only if $<u_R\ u_\phi>$ is positive…

ENERGETICS

The energy flux of the disk is $\rho \; v_R \; (v_\phi^2 /2 \; - GM/R)$.
(We have already made several approximations, ignoring the contributions of thermal pressure and kinetic fluctuations!)

Now,

$$v_R (v_\phi^2 /2 \; - GM/R) = (v_d + u_R) \, [\, (v_K + u_\phi)^2/2 - GM/R]$$

We drop terms that are linear in the u's (zero mean), or of order u^3 or higher (negligibly small). With $v_K^2 = GM/R$, we are left with :

$$v_R (v_\phi^2 /2 \; - GM/R) \; = - v_d \, v_K^2/2 + \langle u_R \, u_\phi \rangle \, v_K \; = -3 v_d \, v_K^2/2$$

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Because $\langle u_R \, u_\phi \rangle = - v_d$!

ENERGETICS

We are studying steady-state conditions,
$M_{dot} = - 2\pi R v_d \Sigma =$ constant, $\int \rho \, dz = \Sigma$.

The integrated energy flux of the disk is thus:

$$-3 \Sigma v_K^2 v_d /2 = +3M_{dot} v_K^2/(4\pi R) = 3GM_{dot}M/(4\pi R^2) .$$

We have found the precise R dependence of the energy flux using turbulence theory! Mechanical energy is not conserved. Taking $\nabla \bullet (\mathit{Flux}) = dE/dt$,

$$dE/dt = - 3GM_{dot}M/(4\pi R^3),$$

the rate at which energy is lost to radiation per unit surface area.

ENERGETICS

But $dE/dt = -2\,\sigma T_s^4$, (local blackbody radiating from top and bottom of the disk). We have thus found the the prediction of local turbulence theory for the surface temperature of the disk,

$$T_s^4 = 3GM_{dot}M/(8\pi\,\sigma\,R^3),$$

A classical result first obtained in viscous disk theory (Lynden-Bell 1969).

TURBULENCE

1) Turbulence theory can be predictive (key assumptions: (energy is dominated by gravity and rotation, existence of correlated fluctuations). But …

2) Why should the u_R and u_ϕ be correlated? And what determines the amplitude of their correlation?

3) MHD seems to be at the heart of turbulent disk theory. The classical results remain valid, however, even though they were derived without magnetic fields…but MHD goes well beyond viscous disk theory.

Why must u_R and u_ϕ be correlated in shear turbulence?

The answer is found in the *exact* equation

$\partial E/\partial t + \nabla \bullet \boldsymbol{F} = -\ \rho\ u_R\ u_\phi\ \ d\Omega/d \ln R\ \ -\ Q_{--},$

Where Ω is v_K/R, *E* is the energy density of fluctuations, ***F*** the associated energy flux, and Q_{--} is dissipation.

This tells us something *very interesting:*

$-d\Omega/dR$ is the free energy source for the fluctuations, and on average, $\langle u_R\ \ u_\phi \rangle$ *must* be > 0 for free energy extraction.

Problem : to ensure $\langle u_R u_\phi \rangle$ is positive.

The question of the stability of Keplerian disks has been *very* controversial. Does large Re differential rotation behave like large Re *Cartesian* shear, breaking down into turbulence? The answer appears to be NO: Keplerian disks are stable, shear layers are not. The epicyclic response decorrelates u_R and u_ϕ, *unless* $\kappa^2 \ll d\Omega^2/d \ln R$.

Be especially skeptical of what you read in older textbooks regarding claims of nonlinear instability in rotating flows.

The Epicyclic Frequency κ

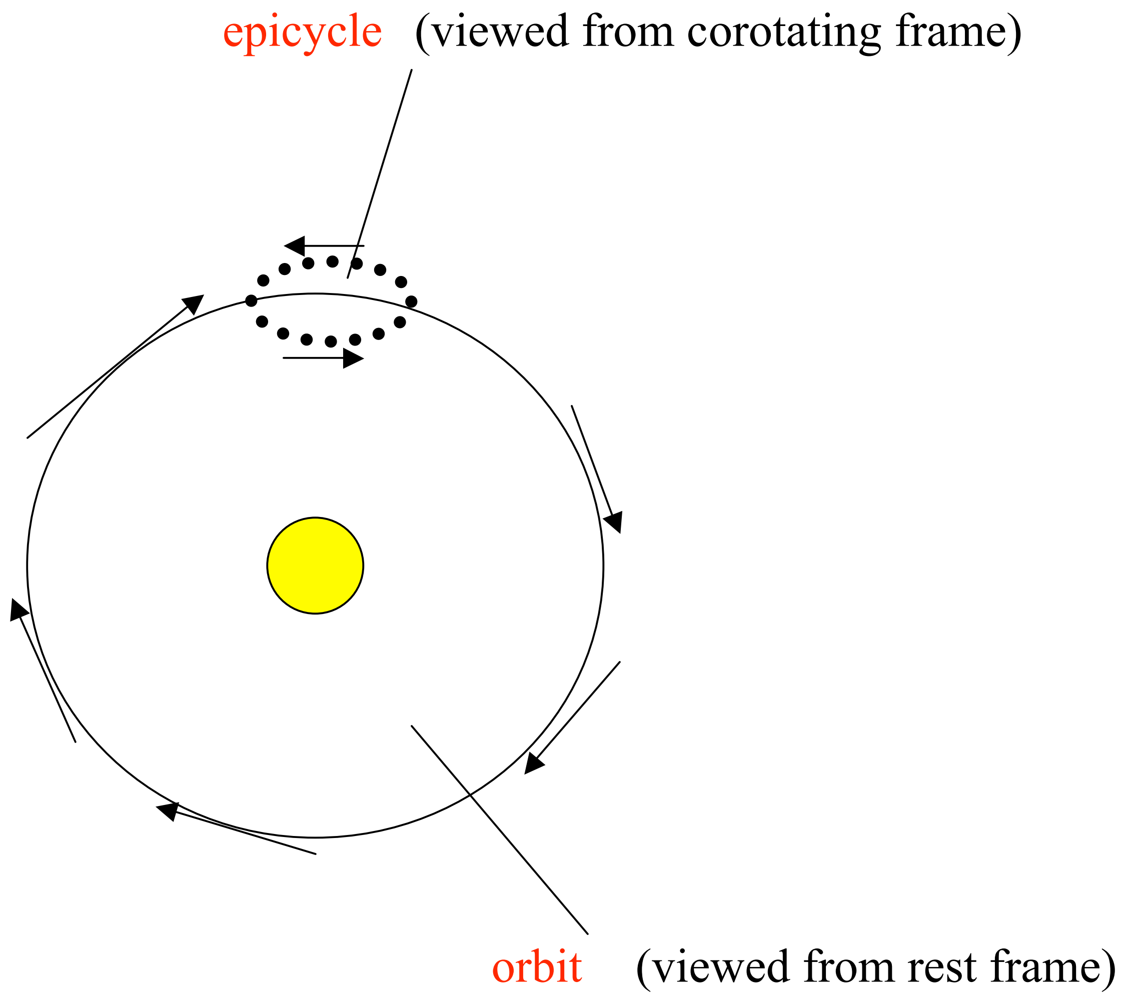
A fluid in circular rotation, when disturbed, oscillates about its equilibrium orbit with an easily calculable frequency, κ:

$$\kappa^2 = 4\Omega^2 + \frac{d\Omega^2}{d\ln R} = \frac{1}{R^3}\frac{d(R^4\Omega^2)}{dR}$$

For elliptic Keplerian orbits this is Ω^2, but the above is more general. When $\kappa^2<0$, the rotation profile is unstable (Rayleigh Criterion).

This corresponds to the specific angular momentum decreasing outwards.

EPICYCLIC MOTION:



BEAR IN MIND:

1) No local or global numerical simulation of a Keplerian disk has ever shown a hint of instability, while simulations do show a breakdown into turbulence for shear layers.

2) The small amplitude response of a Keplerian disk is a stable wave. The small amplitude response of a shear layer is neutral; hence nonlinearities essential.

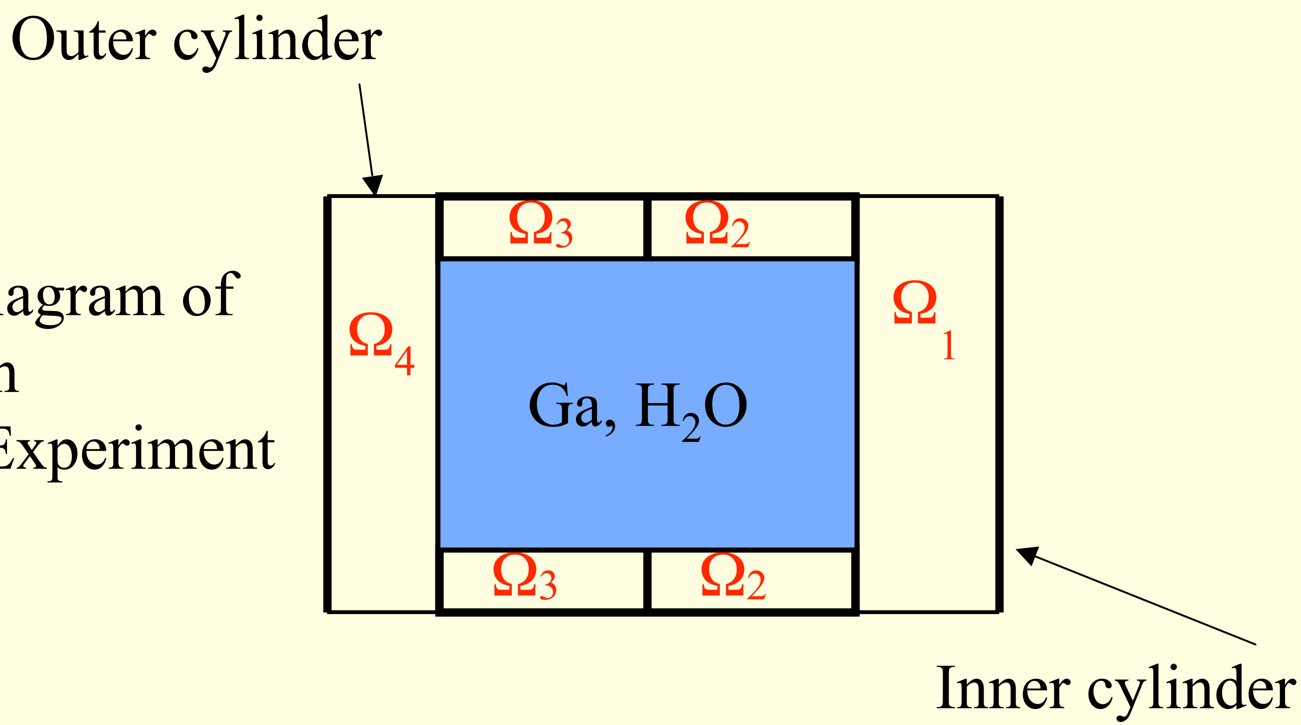
3) Keplerian disks *driven* into turbulence by midplane thermal heating have $\langle u_R \, u_\phi \rangle$ less than zero!

4) Exact solution exists: nonlinear, three-dimensional, and time-dependent for a Keplerian that is stable, and reproducable in simulations (Balbus 2006, in press.)

A RECENT RESULT:

A Couette flow experiment with water conducted by Hantao Ji at the Princeton Plasma Physics laboratory has achieved the highest laboratory Re to date: 2×10^6. The flow is perfectly stable, and $<u_R \, u_\phi>$ is consistent with zero, $<< u^2$.

Schematic Diagram of
The Princeton
MRI/Hydro Experiment



(H_2O: Keplerian profile stable at *Re*~2×10^6,
Ji et al. 2006, *Nature,* in press.)

In Brief:

Lord Rayleigh (1916) showed that a rotating fluid, in which the specific angular momentum “L” increases outward, is stable to axisymmetric disturbances. Coriolis stabilization. When $|2\Omega| > |d\Omega/d(\ln R)|$, nonaxisymmetric disturbances appear also to be stable. $dL^2/dR<0$ true of almost NO astrophysical systems.

But:

It is a remarkable fact that if there is any electrical current flowing in the fluid, the *dynamical* stability properties are dramatically altered. The system is DESTABILIZED, if the rotation rate Ω is faster on inside than on the outside. This is true of almost ALL astrophysical disk systems, which are prone to the magnetorotational instability.

Electric and Magnetic Fields in Fluids

$$E + v_e \times B = 0$$

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$$\frac{\partial B}{\partial t} = \nabla \times (v_e \times B)$$

$$\frac{DB}{Dt} = (B \bullet \nabla) v_e$$

Magnetic fields have a pressure force and a tension force:

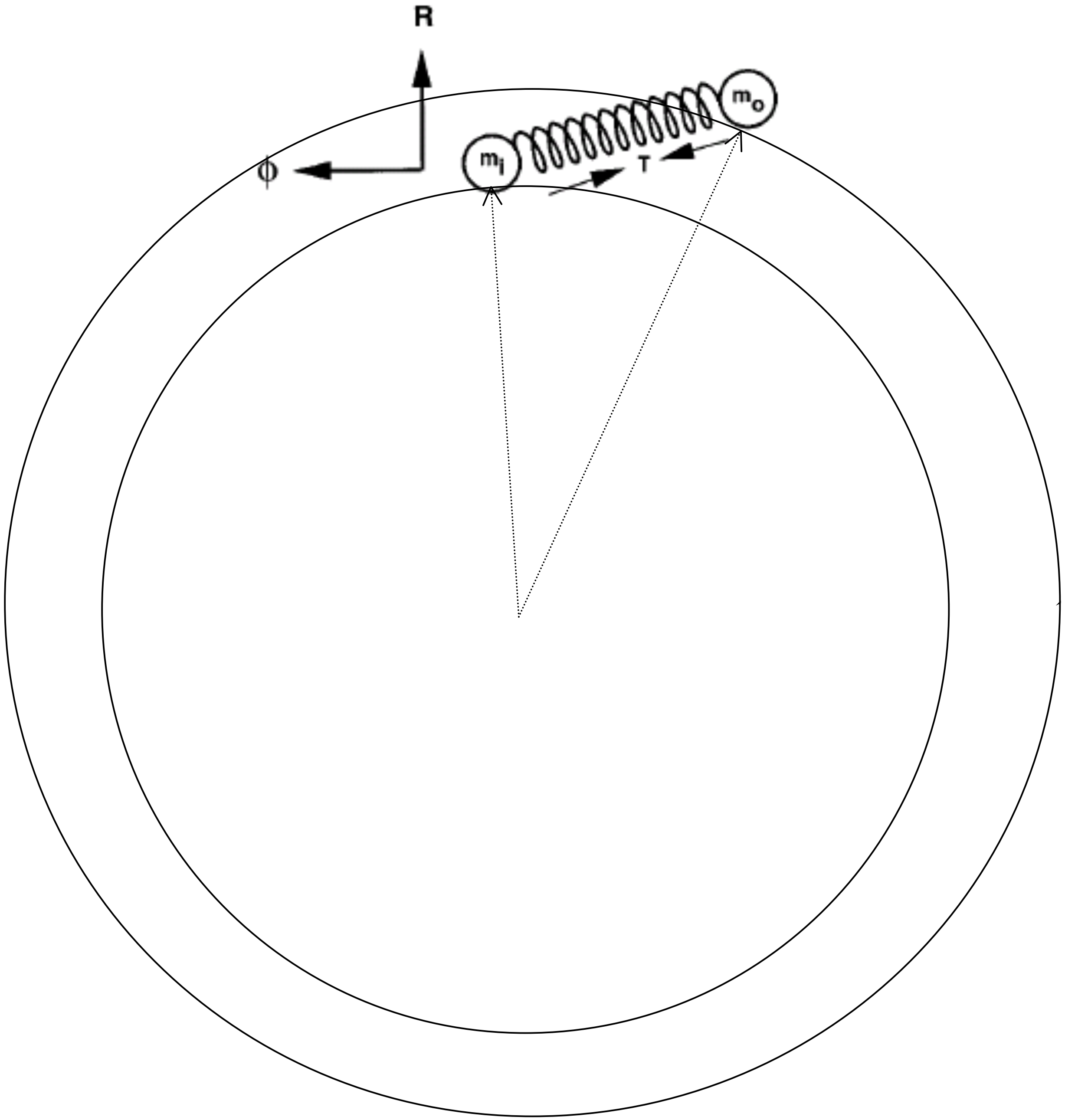
$$F(mag) = -\frac{1}{2\mu_0}\nabla B^2 + \frac{1}{\mu_0}(B \bullet \nabla)B$$

pressure

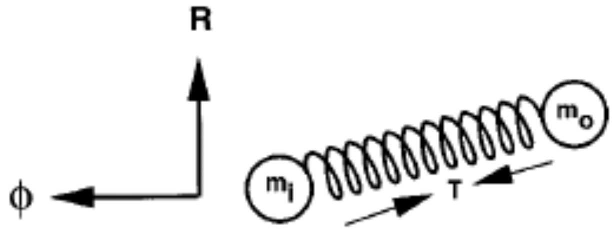
tension

The pressure force is unimportant when gas pressure dominates, but the tension introduces new degrees of freedom. Combination of magnetic tension and frozen-in behavior leads to existence of shear waves, rather like a solid.

In fact, “masses and springs” is not a bad analogy at all…



R
ϕ
m_i
m_o
T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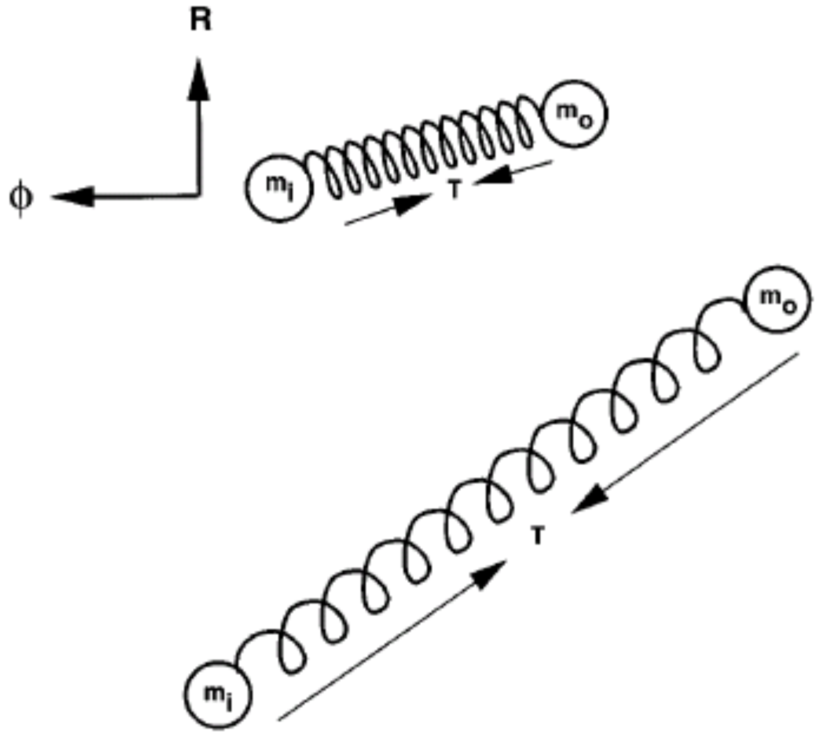
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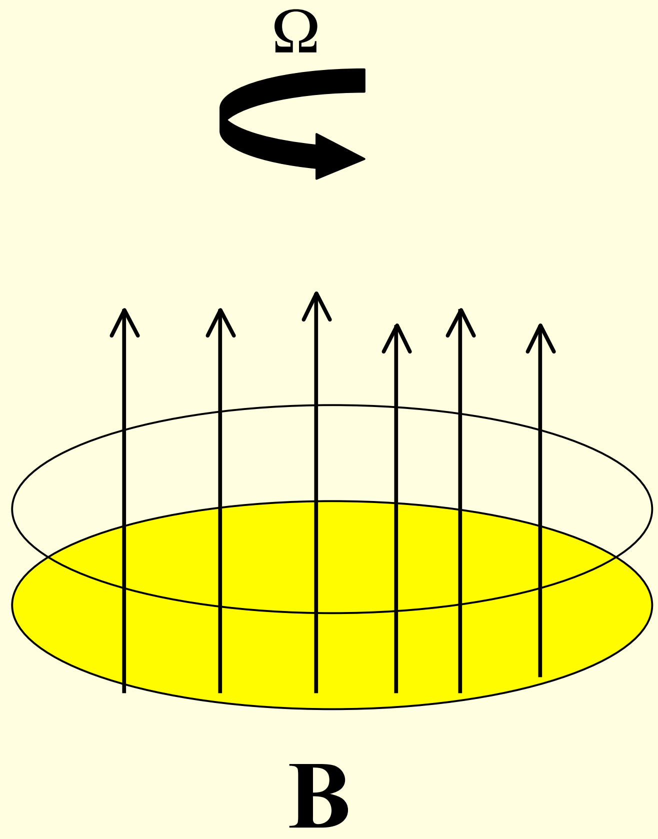
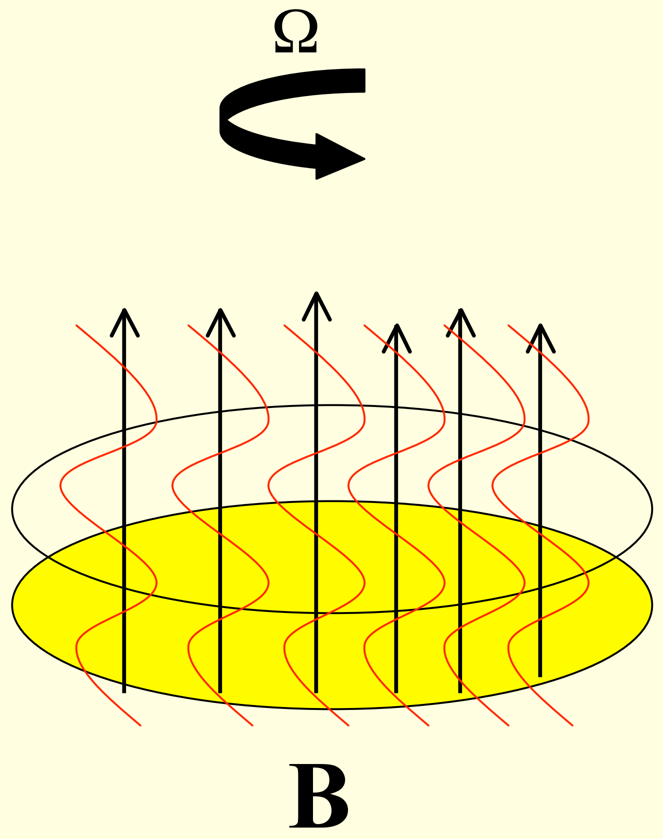
m_o
T

Equilibrium



Perturbed



FUNDAMENTAL EQUATIONS

$$\rho \frac{D\mathbf{v}}{Dt} = -\nabla P - \nabla\Phi - \frac{1}{2\mu_0}\nabla B^2 + \frac{1}{\mu_0}(B \bullet \nabla)B$$

$$\frac{D\ln\rho}{Dt} + \nabla \bullet \mathbf{v} = 0$$

$$\frac{D\mathbf{B}}{Dt} = (\mathbf{B} \bullet \nabla)\mathbf{v}$$

$$P = K\rho^{\gamma}$$

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Equation of motion

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Mass conservation

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← Induction equation

$$P = K\rho^{\gamma}$$

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Energetics

Solutions for small disturbances:

$$\omega^2 = k_z^2 v_A^2$$

$$\omega^2 = \kappa^2$$

$$\omega^4 - \omega^2(\kappa^2 + 2k_z^2 v_A^2) + k_z^2 v_A^2 (k_z^2 v_A^2 + d\Omega^2 / d\ln R) = 0$$

The final term in the MRI dispersion relation is the product of the azimuthal and radial forces acting on a fluid element. There is instability if:

$$k_z^2 v_A^2 + d\Omega^2 / d\ln R < 0$$

This is the magnetorotational instability.

Consider next a rotating system with a magnetic field in the z direction:

$$\omega^2 = k_z^2 v_A^2$$

Alfvén waves

$$\omega^2 = \kappa^2$$

$$\omega^4 - \omega^2(\kappa^2 + 2k_z^2 v_A^2) + k_z^2 v_A^2 (k_z^2 v_A^2 + d\Omega^2 / d\ln R) = 0$$

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Inertial waves

$$\omega^4 - \omega^2(\kappa^2 + 2k_z^2 v_A^2) + k_z^2 v_A^2 (k_z^2 v_A^2 + d\Omega^2 / d\ln R) = 0$$

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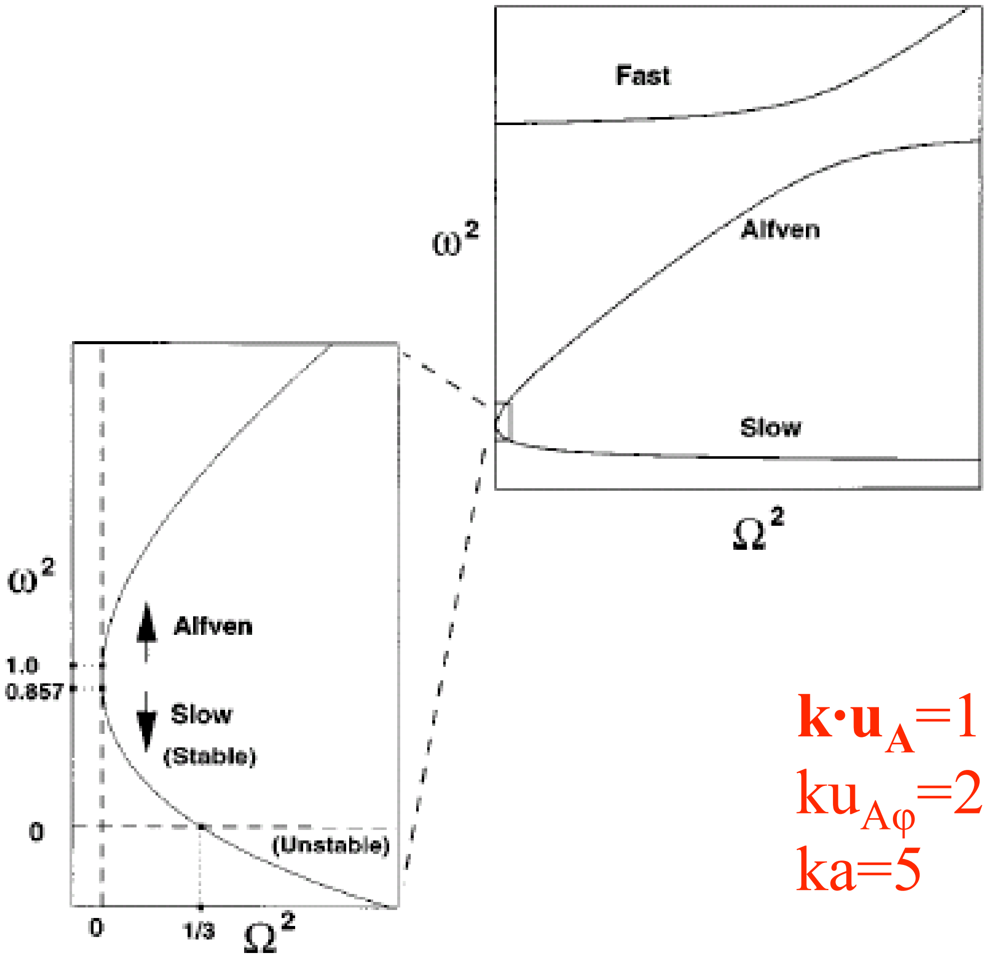
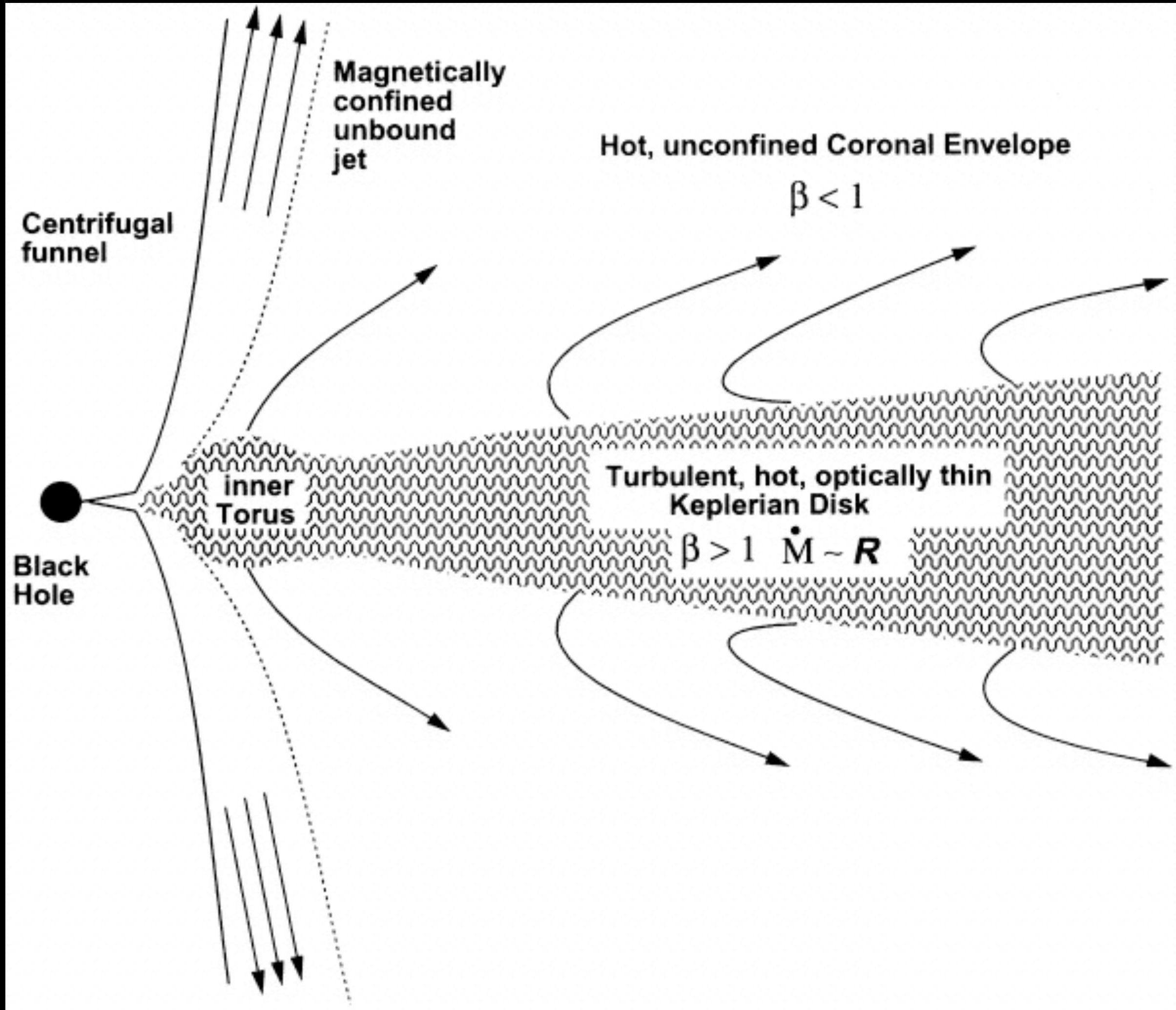


$\mathbf{k \cdot u_A}=1$
$ku_{A\varphi}=2$
$ka=5$

disk and jet

by *John Hawley*


Magnetically
confined
unbound
jet
Hot, unconfined Coronal Envelope
$\beta < 1$
Centrifugal
funnel
inner
Torus
Turbulent, hot, optically thin
Keplerian Disk
$\beta > 1$ $\dot{M} \sim R$
Black
Hole

SUMMARY

1. Magnetic fields profoundly affect the stability of magnetized gas, even when the field is highly subthermal.

2. Adverse free energy gradients (angular velocity, temperature) become sources of instability. Classical criteria require adverse angular momentum (Rayleigh) or entropy (Schwarzschild) gradient.

3. MRI is most widely applicable instability of this type, requiring little more than a magnetic field and $d\Omega/dR < 0$. Accretion disks, stars, and possibly geodynamos are all venues.

SUMMARY

4. The MRI transfers angular momentum outward, in both Eulerian and Langrangian sense.   Ideal for disk accretion.

5. In protostellar disks, departures from ideal MHD are key, critical to our understanding of disk structure and eruptive phenomena.