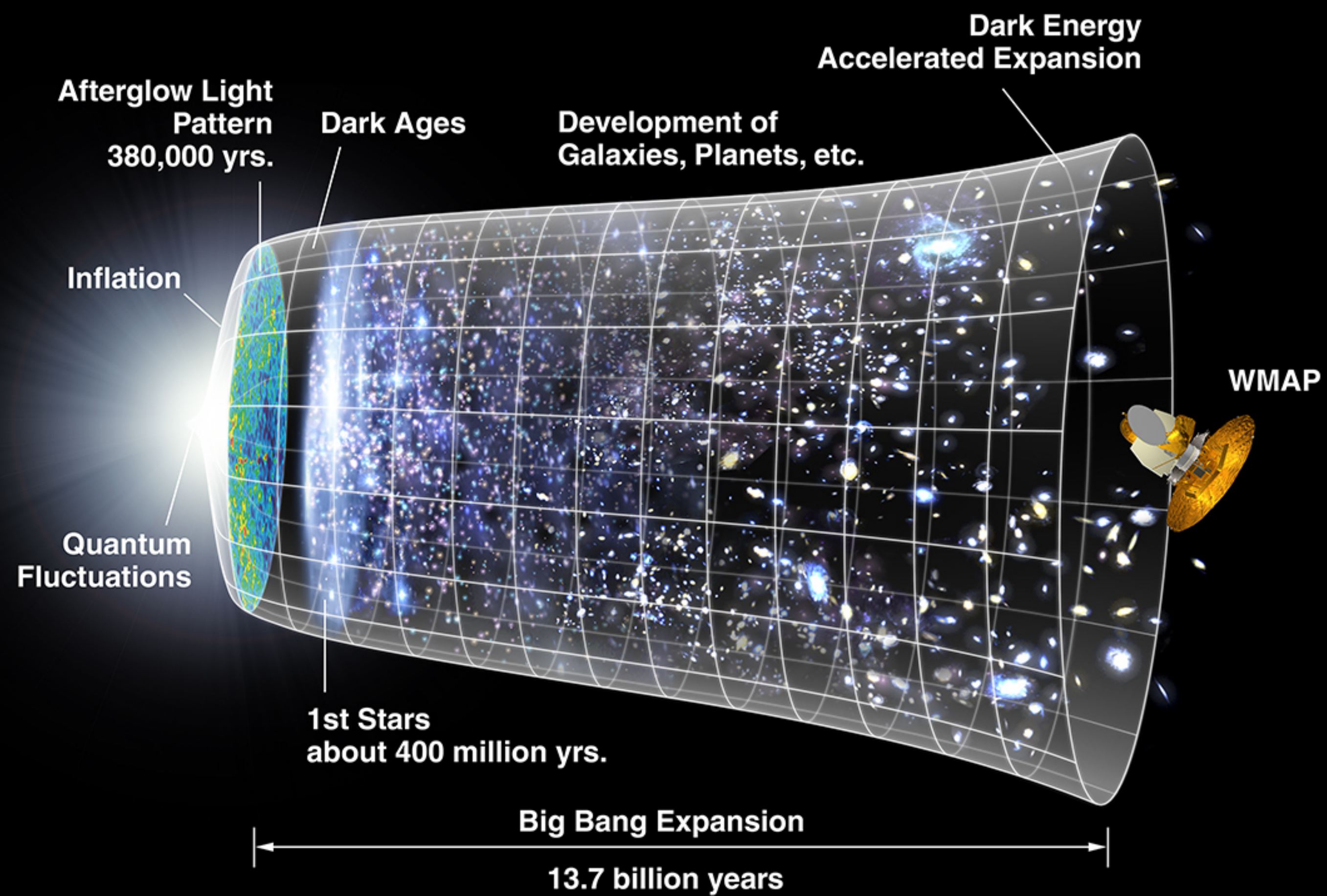
A brief thermal history of the Universe

Martin Kunz
Université de Genève

Overview

- **Equilibrium description**
  - Distribution functions
  - Neutrino decoupling and the neutrino background
  - Photon decoupling, recombination and the CMB
  - BBN and light element abundances
  - Comparison to observations
- **Basics of Boltzmann equation**
  - hot and cold relics
- **Summary**

Brief history of the Universe



Equilibrium distributions

Short-range interactions maintaining thermodynamic equilibrium:

$$f(k,t)d^3k = \frac{g}{(2\pi)^3}\Big(\exp[(E-\mu)/T] \pm 1\Big)^{-1} d^3k$$

$E = \sqrt{k^2 + m^2}$, T temperature, μ chem. pot.

$$n = \int f(k) d^3k$$ number density

$$\rho = \int E(k) f(k) d^3k$$ energy density

$$p = \int \frac{|k|^2}{3E(k)} f(k) d^3k$$ pressure

Relativistic species, m << T

Crank handle, using m = μ = 0 -> E ~ k
(use x=E/T as integration variable)

$$n_B = T^3 \frac{g\zeta(3)}{\pi^2} \quad n_F = \frac{3}{4} n_B$$

$$\rho_B = T^4 \frac{g}{30} \pi^2 \quad \rho_F = \frac{7}{8} \rho_B$$

with $\rho_\gamma \sim a^{-4}$ => $T_\gamma \sim 1/a$
-> ***expanding universe cools down***

-> Stefan-Boltzmann law
$\rho_\gamma \sim T^4$

$$p = \frac{\rho}{3}$$

-> $w_{rad} = p_{rad}/\rho_{rad} = 1/3$

Massive species, m >> T

Expand $E = \sqrt{k^2 + m^2} = m\sqrt{1 + k^2/m^2} \approx m + k^2/(2m)$
and neglect +/-1 wrt exp(m/T)

$$n = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-(m-\mu)/T}$$

$$\rho = mn + \frac{3}{2}nT$$ -> E_{kin} / particle: $$E_{\rm kin} = \frac{3}{2}k_B T$$

$$p = nT \ll \rho$$

Massive particles are suppressed by Boltzmann factor exp(-m/T), so they will quickly drop out of thermal equilibrium when T < m -> 'freeze out' -> effective μ

Multiple relativistic species

If we have several species at different temperatures:

$$\rho_R = \frac{T_\gamma^4}{30}\pi^2 g_* \qquad g_* = \sum_{i\in B} g_i \left(\frac{T_i}{T_\gamma}\right)^4 + \frac{7}{8}\sum_{j\in F} g_j \left(\frac{T_j}{T_\gamma}\right)^4$$

Entropy density: $s = \dfrac{\rho + p}{T} \propto T^3 \rightarrow d(sa^3)/dt = 0$

(use f and $\dot\rho + 3H(\rho+p) = 0$)

$$s = \frac{2\pi^2}{45} g_{*S} T_\gamma^3 \qquad g_{*S} = \sum_{i\in B} g_i \left(\frac{T_i}{T_\gamma}\right)^3 + \frac{7}{8}\sum_{j\in F} g_j \left(\frac{T_j}{T_\gamma}\right)^3$$

- $T_\gamma \propto g_{*S}^{-1/3} a^{-1}$

Now we are ready to study particle evolution in the early universe!

Neutrino decoupling

Interaction rate: Γ — species in equil.: Γ >> H

Expansion rate: H — species decoupled: Γ << H

$$\Gamma(T) = n(T)\langle\sigma v\rangle_T \qquad \sigma_F \simeq G_F^2 E^2 \simeq G_F^2 T^2 \qquad \Gamma_F \sim G_F^2 T^5$$

$$H(T) = \sqrt{\frac{8\pi G}{3}}\sqrt{\rho_R} \simeq \frac{5.44}{m_P} T^2 \qquad g_* = 2 + \frac{7}{8}(3\times 2 + 2\times 2)$$

$$\Rightarrow \boxed{\frac{\Gamma_F}{H(T)} \simeq 0.24 T^3 G_F^2 m_P \simeq \left(\frac{T}{1\mathrm{MeV}}\right)^3}$$

Neutrinos decouple when temperature drops below ~ 1 MeV because their interactions become too weak.

Temperature of ν background

Shortly after the neutrinos decouple, we reach T=0.5MeV=m_e and the entropy in electron-positron pairs is transferred to photons but not to the neutrinos. Photon + electron entropy $g_{*S}(Ta)^3$ is separately conserved:

$$g_*(T_{\nu\mathrm{dec}} > T > m_e) = 2 + \frac{7}{8} \times 4 = \frac{11}{2}, \quad g_*(T < m_e) = 2$$

How much are the photons heated by the electron-positron annihilation?

Temperature of ν background

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$$\frac{(aT_\gamma)^3_{\mathrm{after}}}{(aT_\gamma)^3_{\mathrm{before}}} = \frac{(g_*)_{\mathrm{before}}}{(g_*)_{\mathrm{after}}} = \frac{11}{4}$$

Since $(aT_\nu) = (aT_\gamma)_{\mathrm{before}}$ we now have $T_\gamma = (11/4)^{1/3}\, T_\nu$

-> for $T<0.5m_e$: $g_* \sim 3.36$ and $g_{*S} \sim 3.91$ for radiation (γ+ν)

Photon decoupling

e^+/e^- annihilation stopped by baryon asymmetry, remaining electrons and photons stay in thermal contact (Compton scattering) until electrons and protons form neutral hydrogen (recombination). As number of free electrons n_e drops, photons decouple when $\Gamma_\gamma \sim H$.

$$\Gamma_\gamma = n_e \sigma_T, \quad \sigma_T = \frac{8\pi\alpha_{\mathrm{EM}}^2}{3m_e^2}$$

$$n_j = g_j \left(\frac{m_j T}{2\pi}\right)^{3/2} e^{(\mu_j - m_j)/T}, \quad j = e, p, H \qquad g_p = g_e = 2, \; g_H = 4$$

$$\mu_p + \mu_e = \mu_H$$

due to interactions (number conservation)
-> use this to remove μ' s

$$m_p + m_e = m_H + B$$

energy conservation, binding energy: B=13.6eV

Baryon number density: $n_B = n_p + n_H = n_e + n_H \qquad (n_p = n_e)$

Photon decoupling II

We can write $$\frac{n_H}{n_e n_p} \approx \frac{g_H}{g_p g_e}\left(\frac{m_e T}{2\pi}\right)^{-3/2} e^{B/T}$$

introduce $X_e = n_e/n_B$ (fractional ionisation) and notice that (in equilibrium)

$$\frac{1-X_e}{X_e^2} = \frac{n_H n_B}{n_e n_p} = \frac{4\sqrt{2}\zeta(3)}{\sqrt{\pi}} \underbrace{\boxed{\frac{n_B}{n_\gamma}}}_{\eta} \left(\frac{T}{m_e}\right)^{3/2} e^{B/T}$$ **Saha eqn.**

Assume recombination ~ X_e=0.1 and with η ~ 10^{-10} [why?]
=> T_{rec} ~ 0.31 eV, z_{rec} ~ 1300 (why T_{rec} << B?)

Go back to Γ_γ and compare to matter dominated expansion $H=H_0\Omega_m(1+z)^{3/2}$
$\Rightarrow$ T_{dec} ~ 0.26 eV, z_{dec} ~ 1100 -> origin of CMB!

Notice that recombination and photon decoupling are two different processes, although they happen at nearly the same time.

Photons decouple because n_e drops due to recombination (else z_{dec} ~ 40!).

‘equilibrium’ BBN

- T > 1MeV: p and n in equilibrium through weak interactions
- T ~ 1MeV: weak interactions too slow, ν freeze-out
- Light element binding energies: a few MeV

-> when is it favourable to create the light elements?

same game as before: for species with mass number A = #n +#p and charge Z = #p, assumed in equilibrium with p & n

$$n_A = g_A \left(\frac{m_A T}{2\pi}\right)^{3/2} e^{(\mu_A - m_A)/T} = g_A \frac{A^{3/2}}{2^A} \left(\frac{m_N T}{2\pi}\right)^{3(1-A)/2} n_p^Z n_n^{A-Z} e^{B_A/T}$$

where we used again $\mu_A = Z\mu_p + (A-Z)\,\mu_n$. With $X_A = n_A\, A/n_N$ mass fraction:

$$X_A = \ldots = (\mathrm{const}) \left(\frac{T}{m_N}\right)^{3(A-1)/2} X_p^Z X_n^{A-Z} \eta^{A-1} e^{B_A/T}$$

BBN II: NSE

-> System of equations for 'nuclear statistical equilibrium':

$$1 = X_n + X_p + X_2 + X_3 + \ldots$$

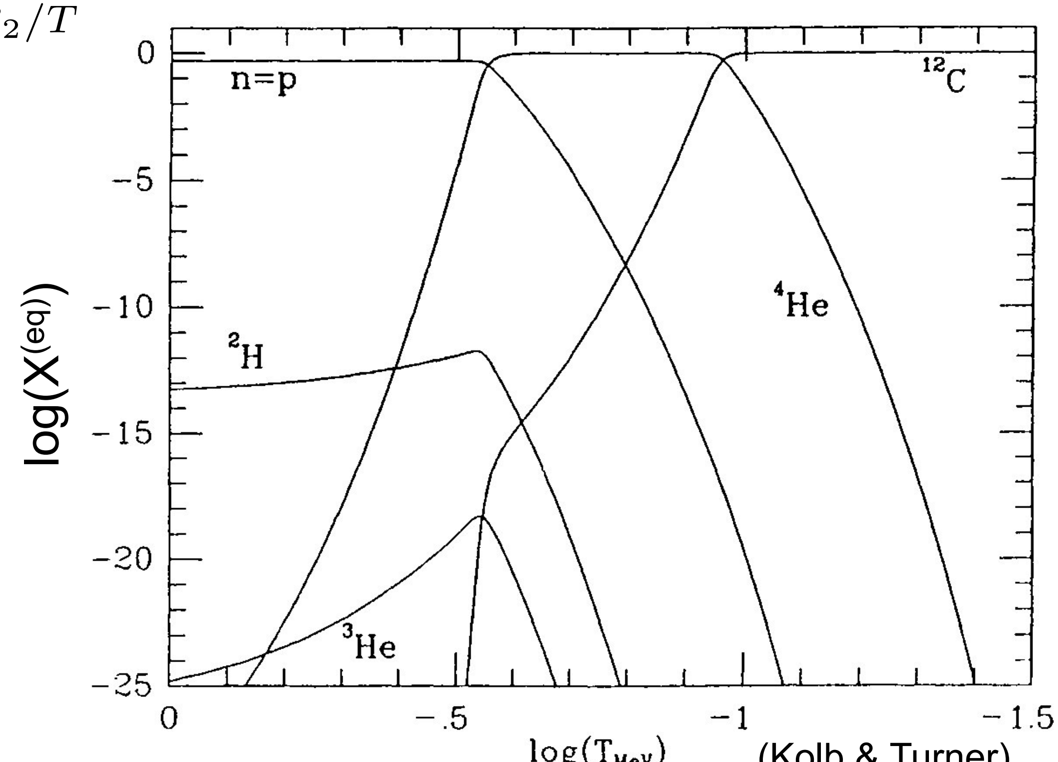
$$X_n/X_p = e^{-Q/T}$$

$$X_2 = C \left(\frac{T}{m_N}\right)^{3/2} X_p X_n \eta e^{B_2/T}$$

$$X_3 = \ldots$$

etc …

- Q = 1.293 MeV
- B[^{2}H] = 2.22 MeV
- B[^{3}H] = 6.92 MeV
- B[^{3}He] = 7.72 MeV
- B[^{4}He] = 28.3 MeV



(Kolb & Turner)

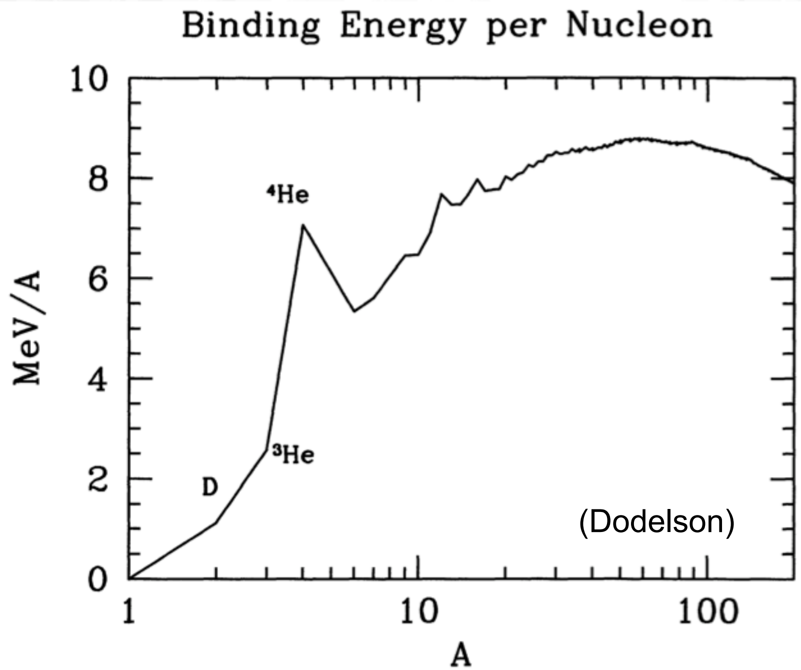
BBN III: actual BBN

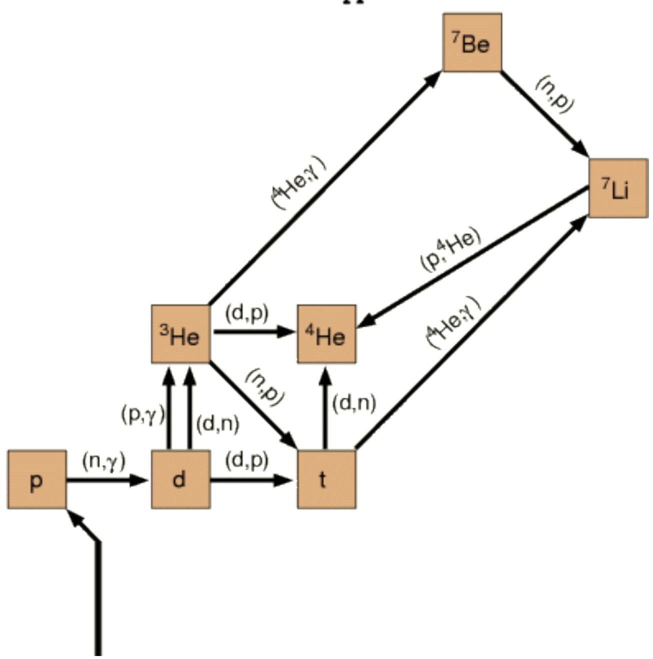
In reality the reactions drop out of equilibrium eventually, and one needs to use the **Boltzmann equation**. Results:

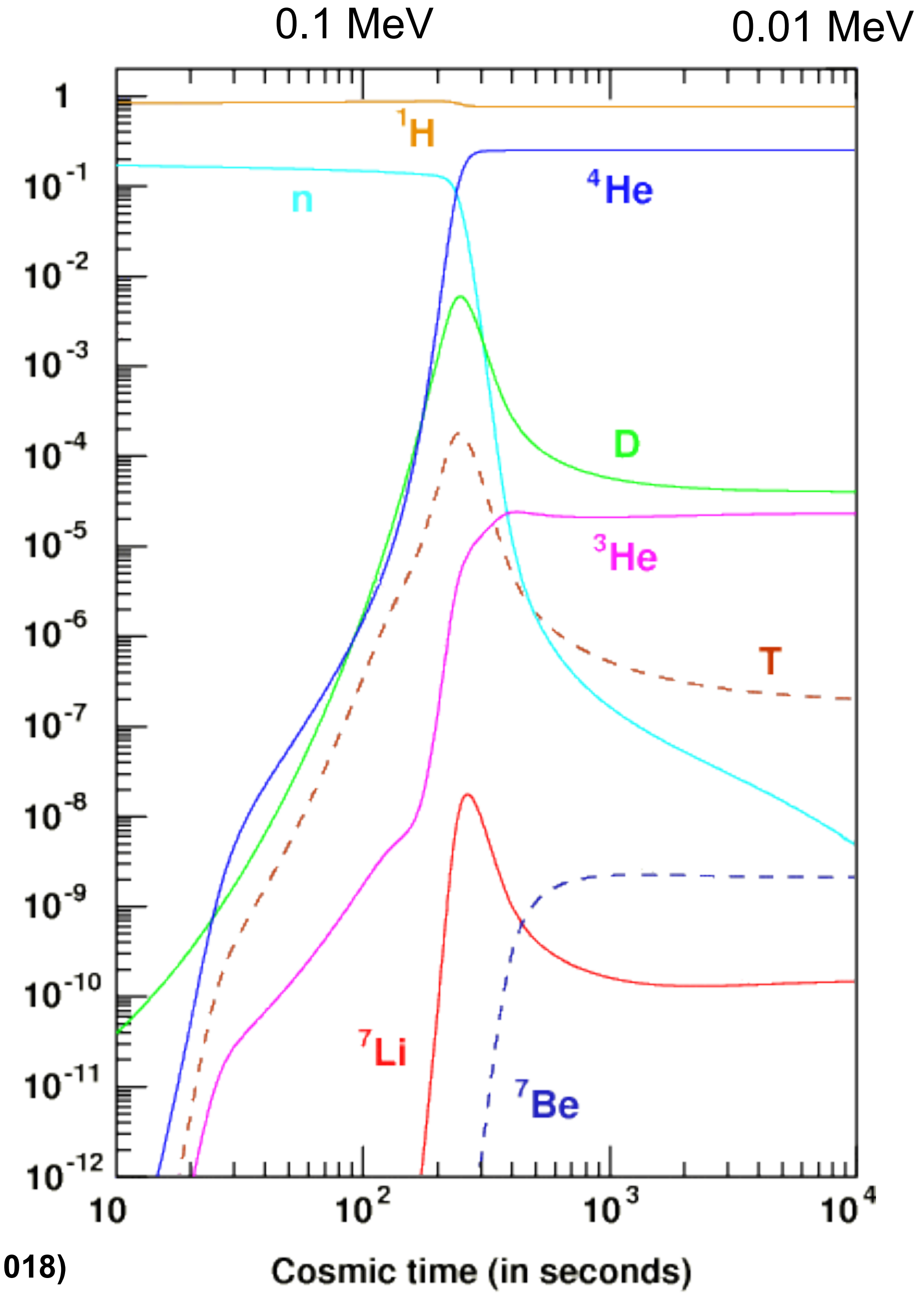
- T ~ 10 MeV+ : equilibrium, $X_n=X_p=1/2$, rest X << 1
- T ~ 1 MeV: n<->p freeze-out, X_n ~ 0.15, X_p ~ 0.85, NSE okay for rest (with X << 1)
- T ~ 0.1 MeV: neutrons decay, n/p~1/8, NSE breaks down because ^{4}He needs Deuterium (^{2}H) which is delayed until 0.07 MeV because of high η and low B_2.
- T ~ 65 keV: now synthesis of ^{4}He can proceed, gets nearly all neutrons that are left:

$$X_{^4\mathrm{He}} = \frac{4n_4}{n_N} = 4\frac{n_n/2}{n_n+n_p} = 2\frac{n_n/n_p}{1+n_n/n_p} = 2\frac{1/8}{1+1/8} \approx 0.22$$

Rest is hydrogen, with some traces of ^{2}H, ^{3}He, ^{7}Li and ^{7}Be.







(figures from A. Weiss, Einstein Online **Vol. 2 (2006), 1018)**

Timeline summary

Energy (γ)	time	event
1 MeV	7s	neutrino freeze-out
0.5 MeV	10s	e^+/e^- annihilation, $T_\gamma \sim 1.4\ T_\nu$
70 keV	3 minutes	BBN, light elements formed
0.77 eV	70' 000 yr	onset of matter domination
0.31 eV	300' 000 yr	recombination
0.26 eV	380' 000 yr	photon decoupling, origin of CMB
0.2 meV	14 Gyr	today

$$\frac{1\ \mathrm{eV}}{k_\mathrm{B}} = \frac{1.602\,176\,53(14)\times 10^{-19}\ \mathrm{J}}{1.380\,6505(24)\times 10^{-23}\ \mathrm{J/K}} = 11\,604.505(20)\ \mathrm{K}.$$

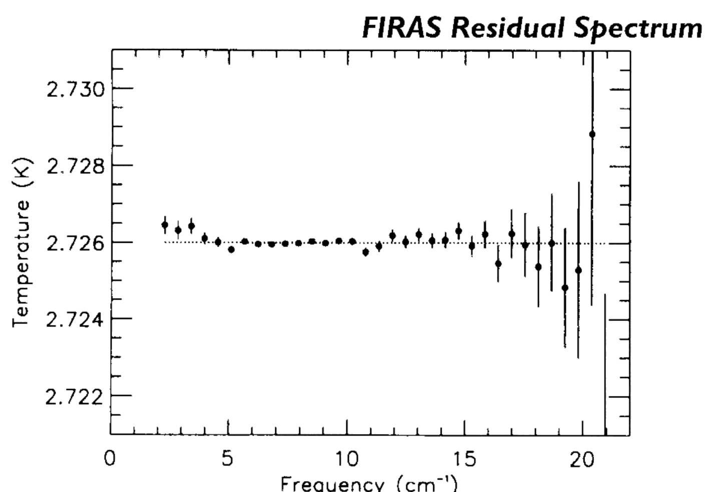
Comparison to observations

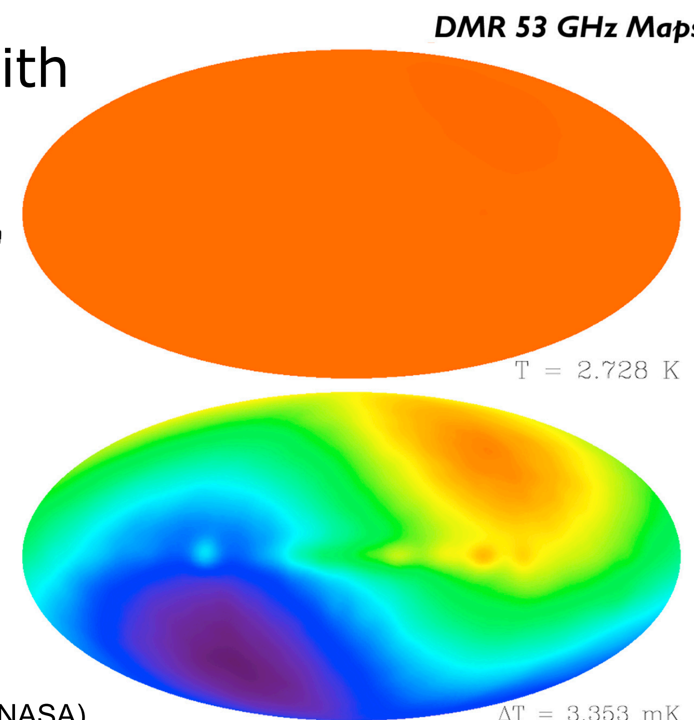
3 classical 'pillars' of big-bang model:

1. Hubble law -> Monday
2. Cosmic Microwave Background
3. BBN and element abundances

Nobel prizes:
1978 CMB discovery
2006 CMB properties
2011 accelerated exp.

CMB: we expect an isotropic radiation with thermal spectrum to fill the universe





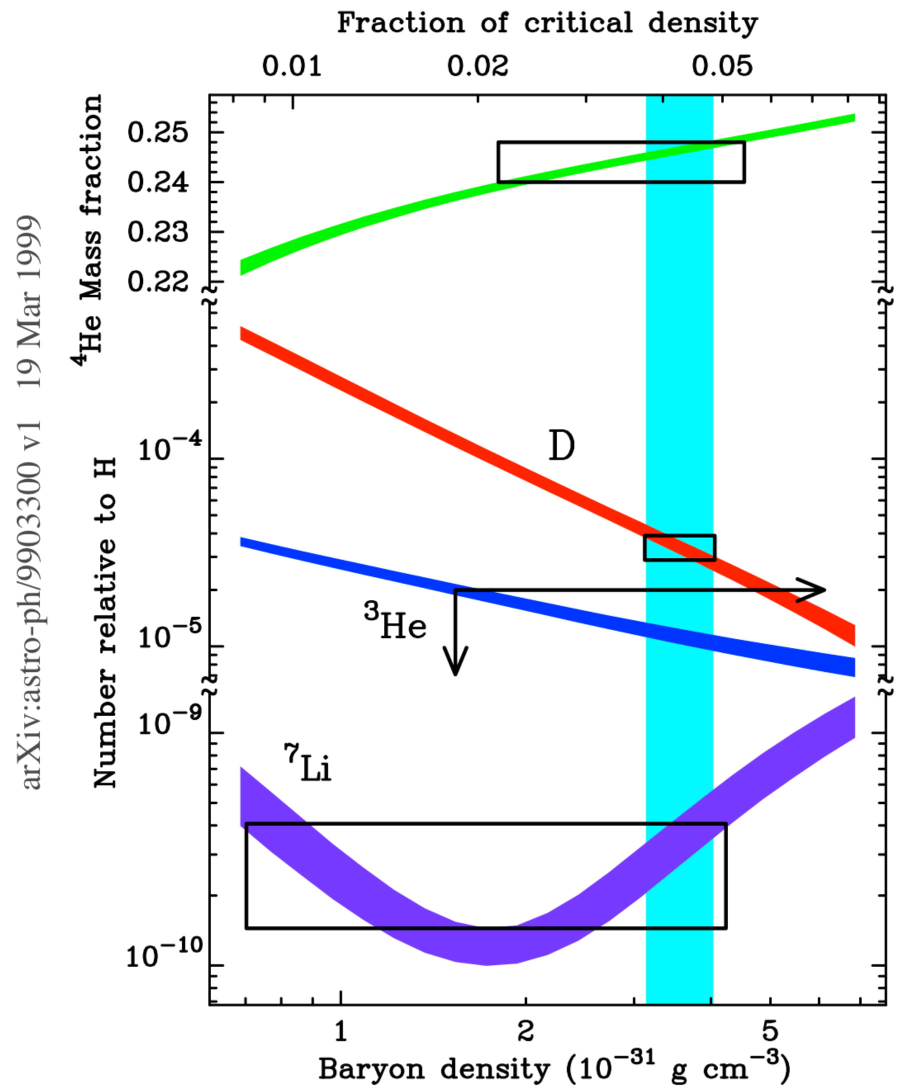
(COBE / NASA)

Comparison II : BBN

BBN tests:

- baryon/photon ratio η
- effective # of relativistic degrees of freedom
- consistency of different abundances

The CMB anisotropies (Valeria) also depend on the baryon abundance (even-odd peak heights). Results are consistent with BBN! $\Omega_b \approx 0.05$
(what is all the rest??!!)



Non-equilibrium treatment

Looking at equilibrium quantities is very useful, but:

- when / how does freeze-out really happen?
- what to do when NSE breaks down
- …

-> Treatment with Boltzmann equation L[f] = C[f]/2

L: Liouville operator ~ d/dt

C: collision operator (→ particle physics!)

relativistic / covariant form: $L = p^\alpha \frac{\partial}{\partial x^\alpha} - \Gamma^\alpha_{\beta\gamma} p^\beta p^\gamma \frac{\partial}{\partial p^\alpha}$

f=f(E,t) -> only α = 0 relevant, p^0=E and Γ from FRW metric

=> $\boxed{L[f] = \left[E \frac{\partial}{\partial t} - \frac{\dot{a}}{a} \overline{p}^2 \frac{\partial}{\partial E} \right] f(E,t)}$

Liouville operator

$$L[f] = \left[E\frac{\partial}{\partial t} - \frac{\dot{a}}{a}\overline{p}^2 \frac{\partial}{\partial E} \right] f(E,t)$$

We only want to know the abundance -> integrate L[f]/E over 3-momentum p to get n:

- first term is just dn/dt
- second term: rewrite in terms of p and integrate by parts (d/dE = E/p d/dp)

$$\boxed{\frac{g}{2\pi^2}\int dp p^2 \frac{L[f]}{E} = \dot{n} + 3Hn}$$

1. no collisions: L[f]=0 -> n ~ a^{-3} as expected!
2. deviations from full equilibrium will be encoded in μ

Collision operator

Roughly: (# of particles 'in') – (# of particles 'out') of phase space volume $d^3p\, d^3x$

For (reversible) scattering of type 1+2 <-> 3+4 and with $d\Pi_i \equiv \frac{g_i}{(2\pi)^3}\frac{d^3p_i}{2E_i}$

$$\frac{g_1}{(2\pi)^3}\int d^3p_1 \frac{C_1[f]}{2E_1} = \int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4)|\mathcal{M}|^2$$
$$[f_3 f_4(1\pm f_1)(1\pm f_2) - f_1 f_2(1\pm f_3)(1\pm f_4)]$$

kinetic equilibrium & low temperature:

$$f(E) \approx e^{(\mu-E)/T}$$
$$[\ldots] \to e^{-(E_1+E_2)/T}\left[e^{(\mu_1+\mu_2)/T} - e^{(\mu_3+\mu_4)/T}\right]$$

with $n_i = e^{\mu_i/T} n_i^{(eq)}$, $\quad n_i^{(eq)} = \frac{g_i}{(2\pi)^3}\int d^3p\, e^{-E_i/T}$ and

$$\langle\sigma v\rangle \equiv \frac{1}{n_1^{(eq)} n_2^{(eq)}}\int d\Pi_1 d\Pi_2 d\Pi_3 d\Pi_4 e^{-(E_1+E_2)/T}(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)|\mathcal{M}|^2$$

$$\Rightarrow \dot{n}_1 + 3Hn_1 = n_1^{(eq)} n_2^{(eq)}\langle\sigma v\rangle\left(\frac{n_3 n_4}{n_3^{(eq)} n_4^{(eq)}} - \frac{n_1 n_2}{n_1^{(eq)} n_2^{(eq)}}\right)$$

Annihilation and freeze-out

Consider annihilation processes: $A + \overline{A} \leftrightarrow Y + \overline{Y}$

assume Y in thermal equilibrium

$$\dot{n}_A + 3Hn_A = \langle \sigma v \rangle \left((n_A^{(eq)})^2 - n_A^2 \right)$$

1) $\langle \sigma v \rangle$ large -> n -> $n^{(eq)}$
2) $\langle \sigma v \rangle$ small -> n ~ a^{-3}

Introduce x=m/T and Y=n/T^3 (Y~n/s, constant for passive evol.)

some algebra…

$$\boxed{\frac{x}{Y_A^{(eq)}} Y_A' = -\frac{\Gamma_A}{H(x)} \left[\left(\frac{Y_A}{Y_A^{(eq)}} \right)^2 - 1 \right]}$$

=> freeze-out governed by Γ/H   ($\Gamma = n^{(eq)} \langle \sigma v \rangle$)

(with $Y^{(eq)} = 0.09$ g (for fermions) if x<<1 and $Y^{(eq)} = 0.16\, g\, x^{3/2} e^{-x}$ if x>>1)

Hot and cold relics

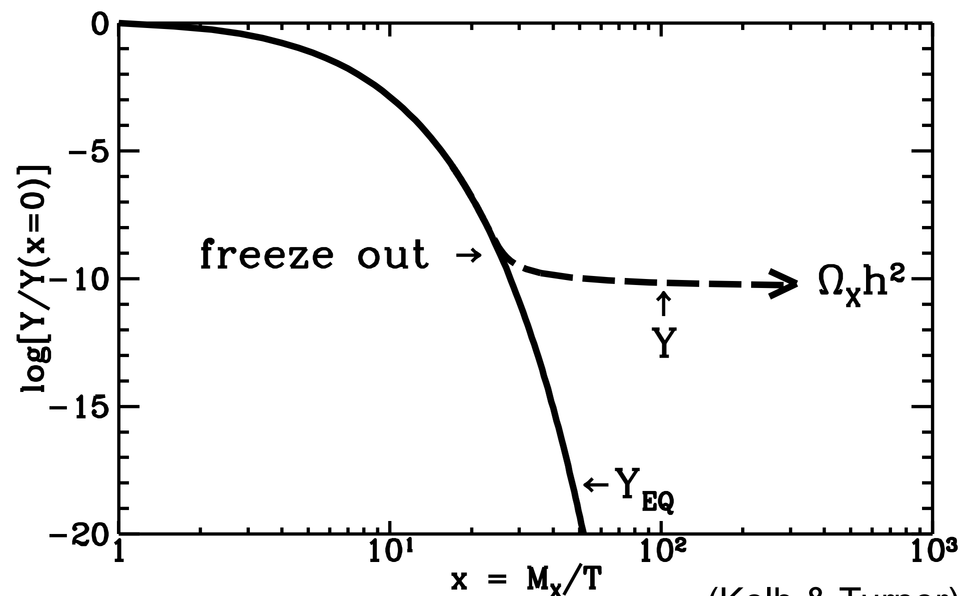
Hot relics: freeze-out when still relativistic (x_f<1)
-> $Y_A(x \rightarrow \infty) = Y_A^{(eq)}(x_f) = 0.278 g_A / g_{*S}(x_f)$

Cold relics: freeze out when x_f >> 1 => Y_A suppressed by $e^{-m/T}$
Abundance generically proportional to 1/σ

We can compute
$\rho_{A,0} = m_A \, n_{A,0}$
$\Omega_A = \rho_{A,0}/\rho_{crit}$

numerically, weak cross-sections lead to Ω ~ 1

-> ***WIMP miracle***



(Kolb & Turner)

Abundance estimate

- Assume $\lambda = \frac{m^3\langle\sigma v\rangle}{H(m)}$ roughly constant -> $\frac{dY}{dx} = -\frac{\lambda}{x^2}\left(Y^2 - Y^{(eq)2}\right)$
- $Y^{(eq)}$ << Y from freeze-out onwards -> $dY/dx \simeq -\lambda Y^2/x^2$
- Integrate from freeze-out to late times -> $1/Y_\infty - 1/Y_f = \lambda/x_f$
- Usually Y_∞ << Y_f -> $Y_\infty \approx x_f/\lambda$
- This depends only linearly on x_f, so take e.g. $x_f = 10$
- particle density scales like a^3 after freeze out

$$\Rightarrow \quad \rho = mn_0 = mn_1\left(\frac{a_1}{a_0}\right)^3 = mY_\infty T_0^3\left(\frac{a_1 T_1}{a_0 T_0}\right)^3 \simeq \frac{mY_\infty T_0^3}{30}$$

(extra factor ~30 from entropy generation in later annihilation processes)

$\Rightarrow$ insert Y_∞ … final: $\Omega_X \sim \left(\frac{x_f}{10}\right)\left(\frac{g_*(m)}{100}\right)^{1/2}\frac{10^{-39}\mathrm{cm}^2}{\langle\sigma v\rangle}$

reasons for decoupling / freeze-out

- **neutrinos** decouple from thermal equilibrium because interactions become too weak ($\sim T^5$ scaling of interaction rate)
- **photons** (CMB) decouple from equilibrium because e^- disappear (recombination)
- **baryons** freeze-out from annihilation because of baryon-antibaryon asymmetry (no more antibaryons to annihilate with)
- **WIMP's** (dark matter) freeze out from because their density drops due to $e^{-m/T}$ Boltzmann factor (if they are WIMP's)

Summary

- **Methods**
  - distribution function f(t,x,v)
  - conservation of entropy
  - full / kinetic equilibrium
  - Boltzmann equation (evolution of f)
- **Results**
  - evolution of particle number, pressure and energy in equilibrium
  - $T \sim 1/a$ (except when g_{*S} changes, e.g. particle annihil.)
  - thermal history, freeze-out of particles when interactions become too slow, WIMPs
  - origin of CMB
  - abundances of light elements

$\left.\begin{array}{l}\text{origin of CMB}\\ \text{abundances of light elements}\end{array}\right\}$ observational “pillars” of the cosmological standard model