Introduction to (homogeneous) cosmology

Martin Kunz
Université de Genève

global outline

funny fluids

- **fundamental notions, the FLRW universe**
  - metric, scale factor, redshift, distances
  - Einstein eqn's, evolution of the universe
  - some resources

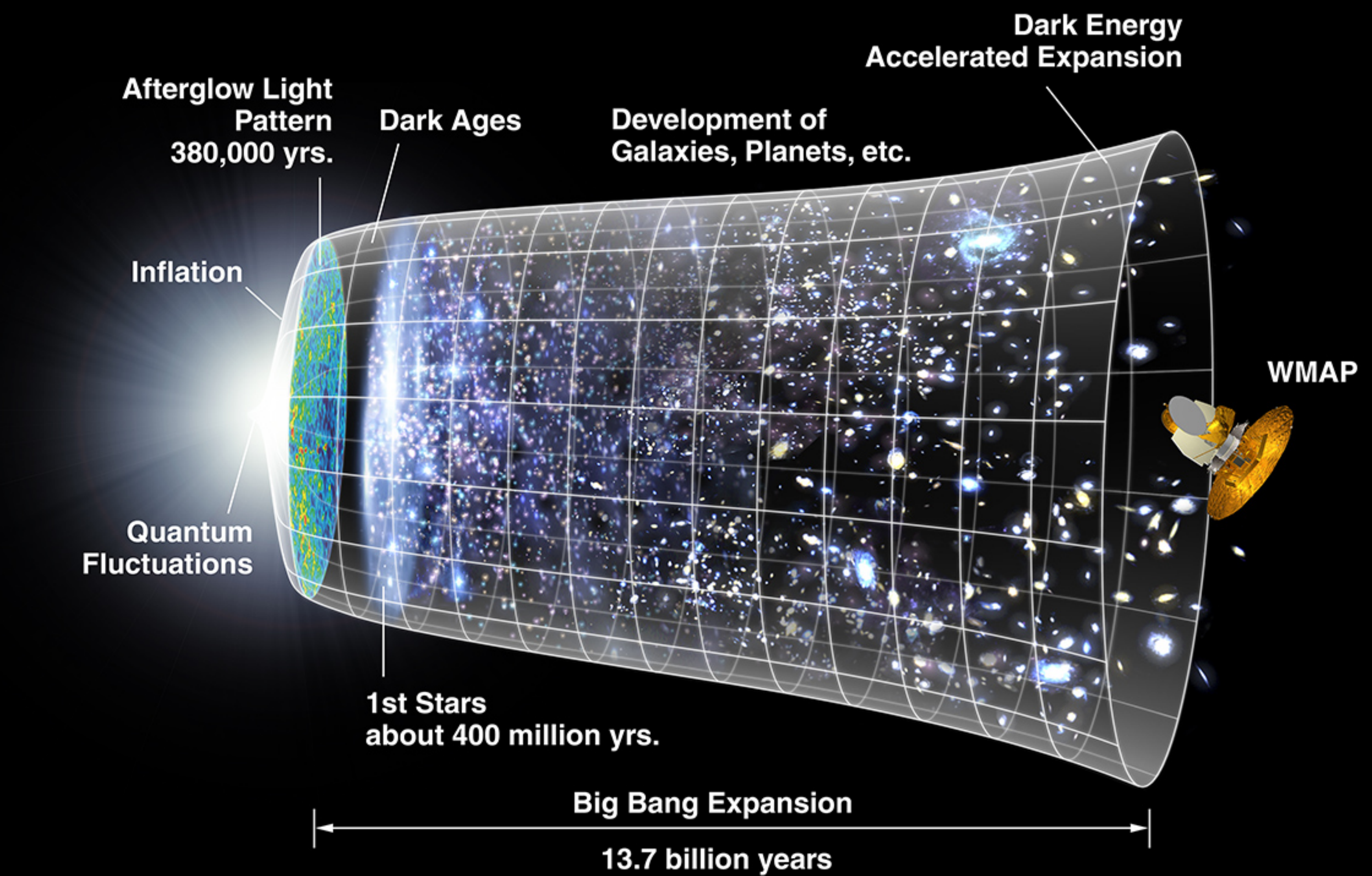
real particles

- **thermal history**
  - relativistic and non-relativistic particles
  - decouplings (neutrinos, photons, WIMP' s)
  - BBN basics

how to lie convincingly

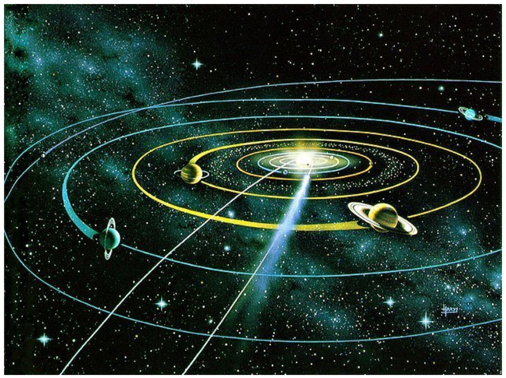
- **statistics & forecasting**
  - basic notions
  - forecasting, Fisher matrix
  - Bayes, parameter estimation, model comparison
  - practical aspects

Brief history of the Universe



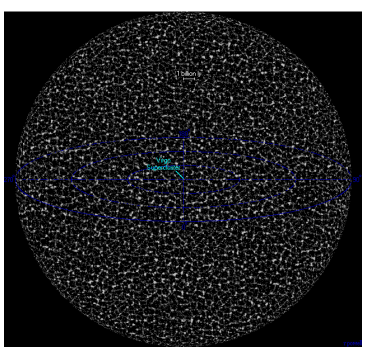
orders of magnitude

cosmology also goes right down to the Planck scale…
… but for now we are more interested in large scales!



solar system:
size: billions of km (10^9 km)
1AU = 1.5×10^8 km
Pluto ~ 40 AU, Voyager 1: 128 AU

galaxies:
size ~ 10 kpc
1pc ≈ 3 light years = 3×10^{13} km
billions of stars (sizes vary!)



(observable) universe
size ~ 10 Gpc (~ 10^{23} km vs $l_P \sim 10^{-38}$ km)
~ 10^{11} galaxies

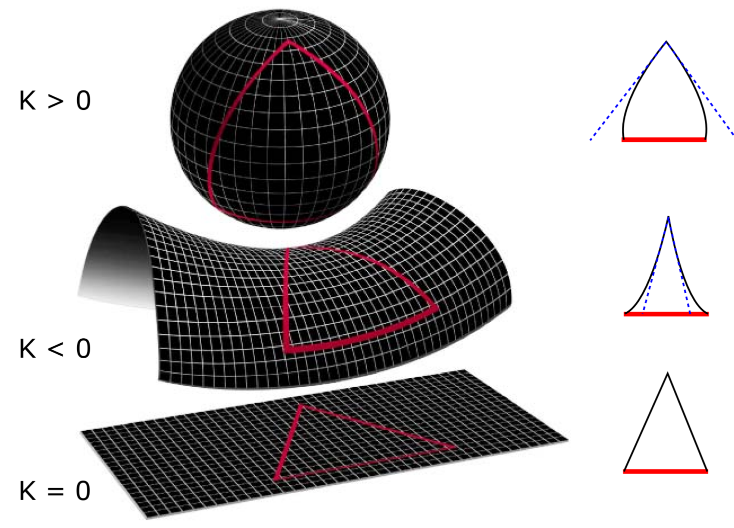
Outline for today

- **metric structure: cosmography**
  - the metric
  - expansion of the universe, redshift and Hubble's law
  - cosmological distances and the age of the universe
- **content and evolution of the universe**
  - Einstein equations and the Bianchi identity
  - the critical density and the Ω's
  - the evolution of the universe
- **short list of key resources**
- **some more things (in case we get bored)**

The metric

- **Cosmological principle**: all observers in the universe are equivalent
- implication: the universe is homogeneous and isotropic
- at least spatially -> **FLRW**
- theorem (differential geometry): spatial sections have constant curvature K(t)

$$ds^2 = dt^2 - \left(\frac{dR^2}{1-KR^2} + R^2 d\Omega\right)$$



(at least for simply connected spaces)

The scale factor

- Maximal symmetry for spatial sections imposes an even stronger constraint: setting R(t) = a(t) r, the line element has the form

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega \right)$$

  where k = ±1 or 0 is a constant

- For this metric, the curves (r,θ,ϕ)=const are geodesics for a 4-velocity u=(1,0,0,0) since $\Gamma^{\mu}{}_{00}$=0 [check!] -> **comoving coordinates**

(geodesic eqn: $\ddot{X}^{\mu} + \Gamma^{\mu}_{\alpha\beta}\dot{X}^{\alpha}\dot{X}^{\beta} = 0$)

The cosmological redshift

Let us consider two galaxies at constant comoving separation ***d***, then for light moving from one to the other($ds^2=0$):

$$d = \int_{t_1}^{t_0} dt/a(t) = \int_{t_1+\delta t_1}^{t_0+\delta t_0} dt/a(t),$$

if δt_1 is e.g. the cycle time at emission, then to keep ***d*** constant we need that for δt_0 :

$$\frac{\delta t_0}{a(t_0)} = \frac{\delta t_1}{a(t_1)} \Rightarrow \frac{\delta t_1}{\delta t_0} = \frac{\nu_0}{\nu_1} = \frac{a(t_1)}{a(t_0)} = \frac{\lambda_1}{\lambda_0}$$

we observe therefore a redshift:

$$z \equiv (\lambda_0 - \lambda_1)/\lambda_1 \Rightarrow \boxed{1 + z = \frac{a(t_0)}{a(t_1)}}$$

The Hubble law

for two galaxies at a fixed **comoving** distance r_0:

physical distance $x(t) = a(t)r_0$

-> apparent motion: $$\frac{dx}{dt} = \dot{a}r_0 = \frac{\dot{a}}{a}x \equiv H_0 x$$

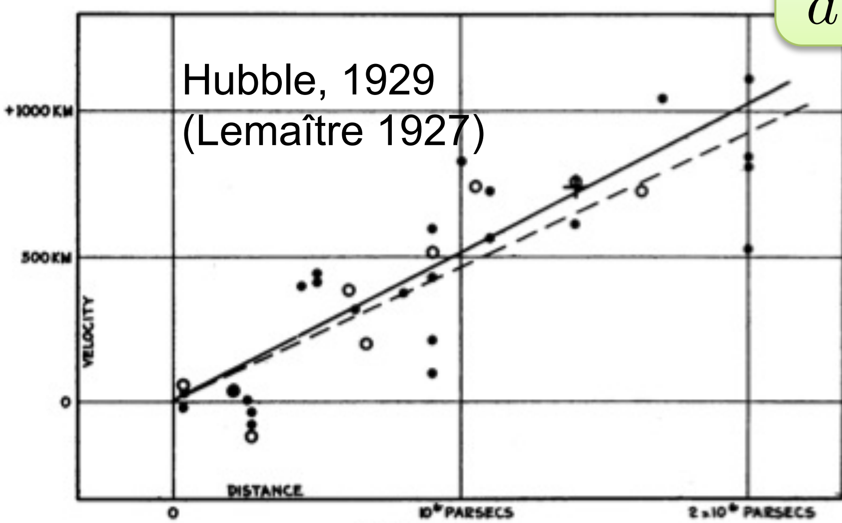
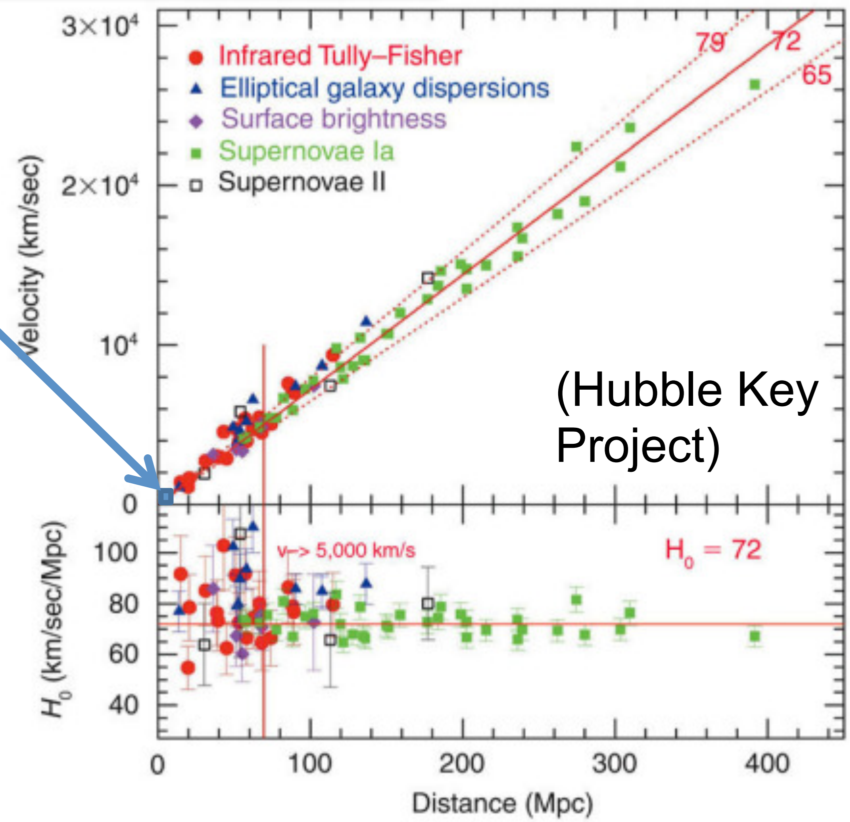


FIGURE 1

Velocity-Distance Relation among Extra-Galactic Nebulae.

Radial velocities, corrected for solar motion, are plotted against distances estimated from involved stars and mean luminosities of nebulae in a cluster. The black discs and full line represent the solution for solar motion using the nebulae individually; the circles and broken line represent the solution combining the nebulae into groups; the cross represents the mean velocity corresponding to the mean distance of 22 nebulae whose distances could not be estimated individually.



some philosophical remarks

- The FLRW metric is just picked ‘by hand’

$$ds^2 = dt^2 - a(t)^2 \left(\frac{dr^2}{1 - \kappa r^2} + r^2 d\Omega \right)$$

- ***This needs to be tested as much as possible!***
- E.g. an even more symmetric possibility would be the de Sitter metric, but observations rule it out!
- We know that the Universe is not exactly FLRW, it’s not entirely clear yet how important this is
- FLRW leads to testable consequences (see the ‘3 pillars’ tomorrow – there are more tests)
- Unfortunately we have only 1 Universe, and we can’t even go everywhere, we can only observe

cosmological distances

simpler to transform the distance variable r to χ:

$$r = S_\kappa(\chi) = \begin{cases} \sin\chi & \kappa = +1 \\ \chi & \kappa = 0 \\ \sinh\chi & \kappa = -1 \end{cases}$$

$$\Rightarrow ds^2 = dt^2 - a^2(t)\left(d\chi^2 + S_\kappa(\chi)^2 d\Omega\right)$$

$$\Rightarrow dV = a_0^2 S_\kappa(\chi)^2 d\Omega d\chi$$ volume element today

we can now *define* a «metric» distance:

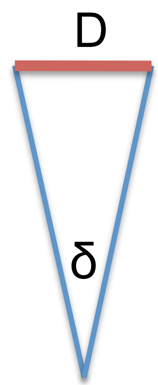
$$\boxed{d_m(\chi) = a_0 S_\kappa(\chi)}$$

$$\chi = \int_{t_1}^{t_0} \frac{dt}{a(t)} = \int_{a_1}^{a_0} \frac{da}{a\dot{a}} = \frac{1}{a_0}\int_0^{z_1} \frac{dz}{H(z)}$$

cosmological distances

but physical distances need to be **observables**!

1) angular diameter distance: object of physical size D observed under angle δ, but photons were emitted at time $t_1 < t_0$:



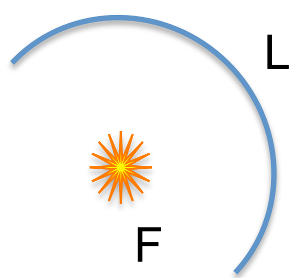
$$D = a(t_1) S_\kappa(\chi)\delta = \frac{a(t_1)}{a_0} a_0 S_\kappa(\chi)\delta \equiv d_A \delta$$

$$\boxed{d_A = \frac{1}{1+z} d_m}$$

2) luminosity distance: consider observed flux F for an object with known intrinsic luminosity L («standard candle»)

$$F \equiv \frac{L}{4\pi d_L^2}$$

surface: $4\pi d_m^2$

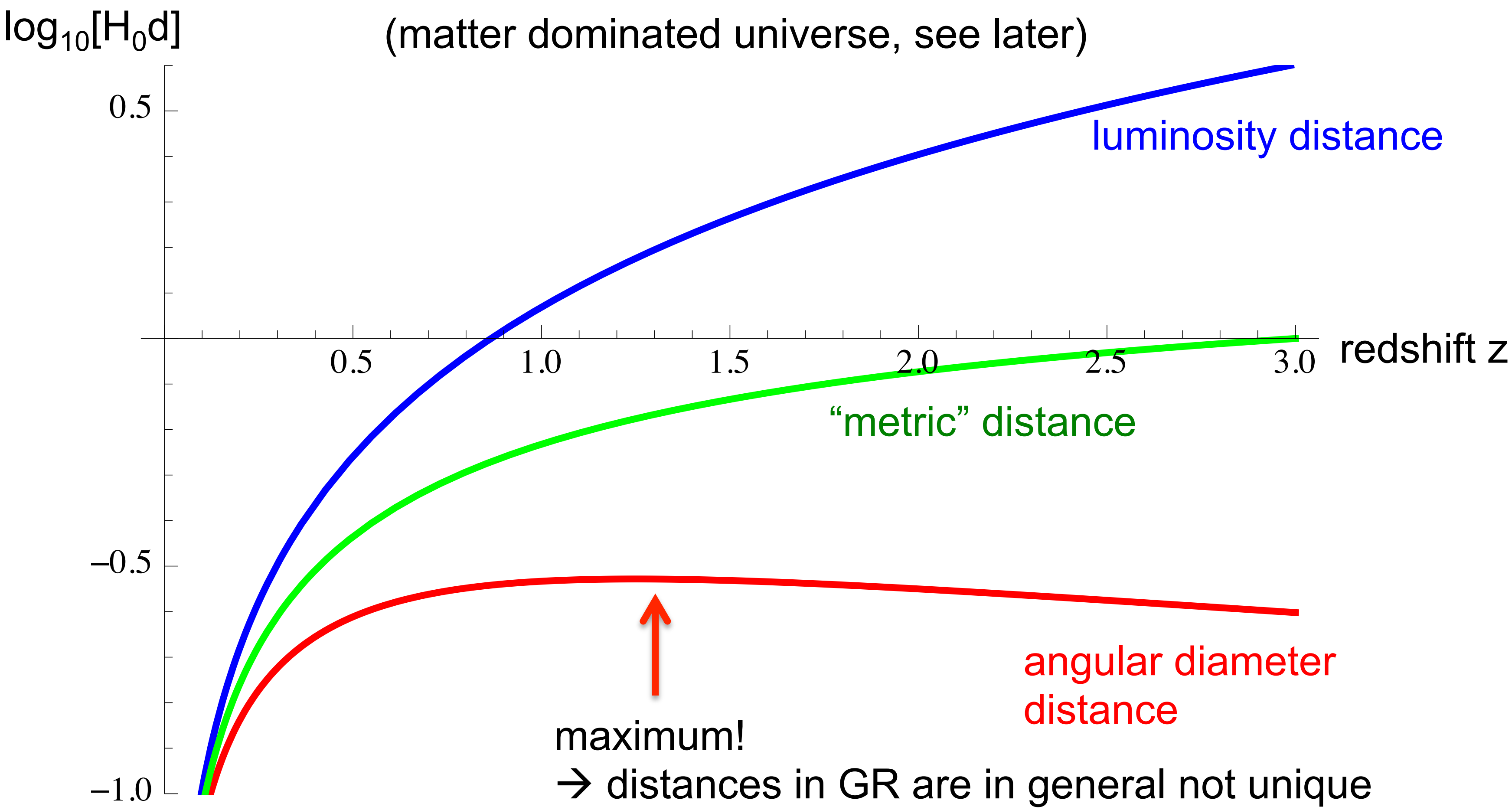


source emitting one photon per second:

1) redshift
2) increased time between arrivals

$$\boxed{d_L = (1+z) d_m}$$

distance example



remark: $d_L = (1+z)^2 \, d_A$ is very general

age of the universe

computing the age of the universe is very straightforward:

$$t_0 = \int_0^{t_0} dt = \int_0^{a_0} \frac{da}{\dot{a}} = \int_0^{a_0} \frac{da}{aH(a)} = \int_0^{\infty} \frac{dz}{H(z)(1+z)}$$

but we need to know the evolution of the scale factor a(t). This in turn depends on the contents of the universe…

cue Einstein: $\underbrace{G_{\mu\nu}}_{\text{geometry}} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N \underbrace{T_{\mu\nu}}_{\text{content}}$

what is in the universe?

- homogeneous and isotropic metric: matter does also have to be distributed in this way
- in **some** coordinate system the energy momentum tensor has the form:

$$T_0^i = 0, \quad T_1^1 = T_2^2 = T_3^3$$

  and the components depend only on time

$$\boxed{T_\mu^\nu = \mathrm{diag}\left(\rho(t), -p(t), -p(t), -p(t)\right)}$$

- the pressure determines the nature of the fluid, p = w ρ:
  – w = 0 : pressureless ‘dust’, ‘matter’
  – w = 1/3 : radiation
  – what is w for $T_{\mu\nu} = \Lambda g_{\mu\nu}$?

the conservation equation

- Bianchi identity (geometric identity for $G_{\mu\nu}$):

$$T^{\mu\nu}_{;\mu} = 0 = G^{\mu\nu}_{;\mu}$$

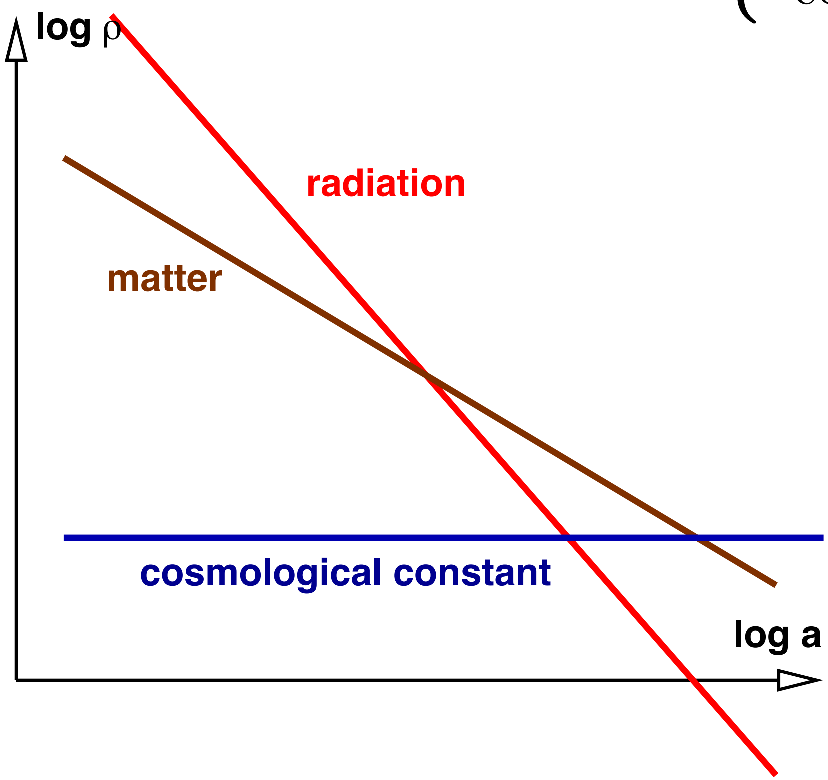
$$T^{\nu}_{0;\nu} = \dot{\rho} + \Gamma^{i}_{i0}(\rho + p) = \boxed{\dot{\rho} + 3\left(\frac{\dot{a}}{a}\right)\underbrace{(\rho + p)}_{(1+w)\rho} = 0}$$

Questions (5 minutes, in groups):

- for a constant w, what is the evolution of ρ(a)?
  (eliminate the variable t from the equation)
- for the three cases w = 0, 1/3, -1, what is ρ(a)?
- does the result make sense?

evolution of the energy densities

$$\rho \propto a(t)^{-3(1+w)} \propto \begin{cases} a(t)^{-3} & \text{for } w = 0 & \text{(matter)} \\ a(t)^{-4} & \text{for } w = 1/3 & \text{(radiation)} \\ \text{const.} & \text{for } w = -1 & \text{(vacuum energy)} \end{cases}$$



dust/matter: dilution through expansion of space

radiation: additional redshift

at early times, the energy density in the universe should have been dominated by radiation

Einstein equations

- we now have all necessary ingredients to compute the **Einstein equations**:
  - **metric**
  - **energy-momentum tensor**

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu}$$

$$R_{\mu\nu} \equiv R^{\alpha}_{\mu\alpha\nu} \qquad R \equiv g^{\mu\nu} R_{\mu\nu}$$

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\nu\beta,\mu} - \Gamma^{\alpha}_{\mu\beta,\nu} + \Gamma^{\delta}_{\nu\beta}\Gamma^{\alpha}_{\mu\delta} - \Gamma^{\delta}_{\mu\beta}\Gamma^{\alpha}_{\nu\delta}$$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right)$$

try to do it yourselves… ☺

Friedmann equations

you should find:

$$R_{00} = -3\frac{\ddot{a}}{a} \qquad R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{\kappa}{a^2}\right] g_{ij}$$

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2}\right]$$

the **space-time** curvature is non-zero even for k=0!

0-0 component:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G_N}{3}\rho$$

sum of ρ from all types of energy

i-i component:

$$2\left(\frac{\ddot{a}}{a}\right) + \left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = -8\pi G_N p$$

Friedmann equations II

three comments:

- you can combine the two equations to find

  $$\left(\frac{\ddot{a}}{a}\right) = -\frac{4\pi G_N}{3}\left(\rho + 3p\right)$$

  -> the expansion is accelerating if $p<-\rho/3$

- the two Einstein equations and the conservation equation are not independent
- there are 3 unknown quantities (ρ, p and a) but only two equations, so one quantity needs to be given (normally p) – as well as the constant k.

the critical density

Friedmann eq. $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G_N}{3}\rho$$

$$H \equiv \left(\frac{\dot{a}}{a}\right) \qquad\qquad \frac{\kappa}{a^2 H^2} = \frac{8\pi G_N \rho}{3H^2} - 1 \equiv \frac{\rho}{\rho_c} - 1 \equiv \Omega - 1$$

$$\Omega(t) > 1 \quad \Rightarrow \quad \kappa > 0 \Rightarrow \textbf{closed} \text{ universe}$$
$$\Omega(t) = 1 \quad \Rightarrow \quad \kappa = 0 \Rightarrow \textbf{flat} \text{ universe}$$
$$\Omega(t) < 1 \quad \Rightarrow \quad \kappa < 0 \Rightarrow \textbf{open} \text{ universe}$$

and: $$\frac{d}{dt}\underbrace{\left(\frac{\Omega - 1}{\kappa}\right)}_{|\Omega - 1| \quad (\kappa \neq 0)} = \frac{d}{dt}\frac{1}{\dot{a}^2} = -2\frac{\ddot{a}}{\dot{a}^3}$$

>0 for expanding universe filled with dust or radiation (and k ≠ 0)
-> the universe becomes “less flat”
-> strange (why?)

‘Ω form’ of Friedmann eq.

Friedmann eq. $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\kappa}{a^2} = \frac{8\pi G_N}{3}\rho$$

evolution of ρ for the «usual» 4 constituents:

- radiation: a^{-4}
- dust: a^{-3}
- curvature: a^{-2} ($H^2 + k/a^2 \sim \rho$)
- cosmological constant: a^0

notation:

$$\Omega_X = \left.\frac{\rho_X}{\rho_c}\right|_{t_0}$$

$$H^2 = H_0^2\left[\frac{8\pi G}{3H_0^2}\rho_0\left(\frac{a}{a_0}\right)^{-n} + \ldots + \frac{\kappa}{H_0^2 a_0^2}\left(\frac{a}{a_0}\right)^{-2}\right]$$

$$H^2 = H_0^2\left[\Omega_r\left(\frac{a}{a_0}\right)^{-4} + \Omega_m\left(\frac{a}{a_0}\right)^{-3} + \Omega_\Lambda + \Omega_\kappa\left(\frac{a}{a_0}\right)^{-2}\right]$$

$$\Omega_r + \Omega_m + \Omega_\Lambda + \Omega_\kappa = 1$$

evolution of the universe I

evolution of ρ for the «usual» 4 constituents:

- radiation: a^{-4}
- dust: a^{-3}
- curvature: a^{-2} ($H^2 + k/a^2 \sim \rho$)
- cosmological constant: a^0

notation:

$$\Omega_X = \left.\frac{\rho_X}{\rho_c}\right|_{t_0}$$

the primordial universe is dominated by radiation – sufficiently early we can neglect the rest!

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \Omega_r \left(\frac{a_0}{a}\right)^4 \Rightarrow a\dot{a} = \text{const.} \Rightarrow a da \propto dt$$

$$\boxed{a(t) \propto t^{1/2}}$$

"radiation-dominated expansion law"

evolution of the universe II

later, the radiation can be neglected (and we set Λ=0)

$$\dot{a}^2 = -\kappa + \frac{C}{a}, \qquad C = H_0 \Omega_0 a_0^3 > 0$$

1) $\kappa = 0 \rightarrow \sqrt{a}da = \sqrt{C}dt \text{ et } a(0) = 0 \Rightarrow \boxed{a(t) \propto t^{2/3}}$

"matter-dominated expansion"

2) $\kappa = 1 \rightarrow \dot{a}^2 = \dfrac{C}{a} - 1$

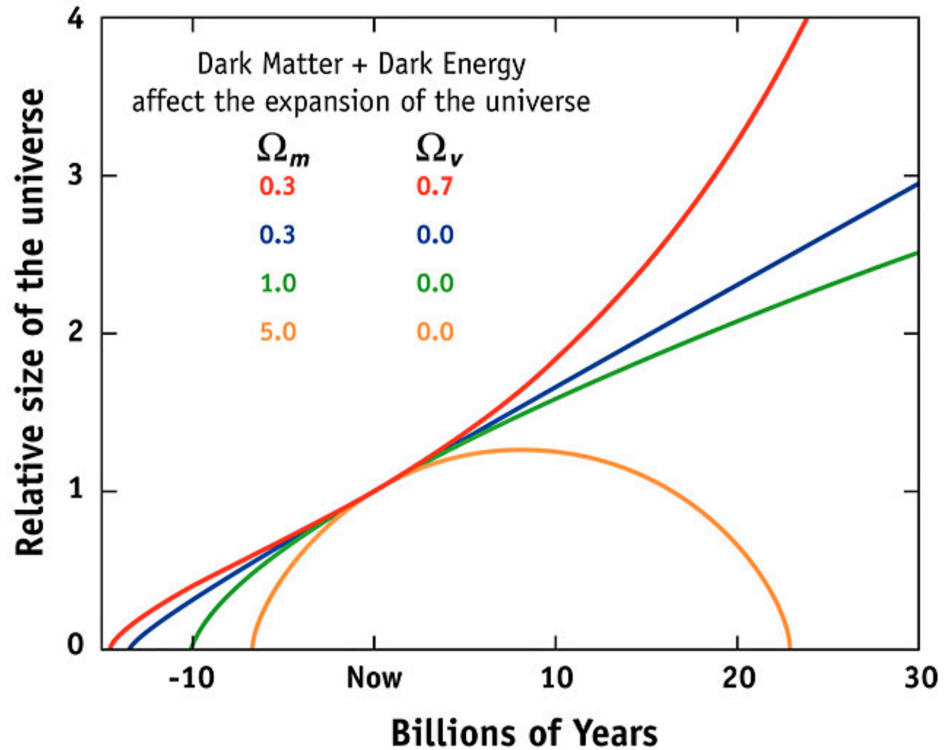
- for a=C we have that $\dot{a} = 0$ and the universe changes from expansion to contraction/collapse
- the substitution $a = C\sin^2\theta$ leads to $2C\sin^2\theta d\theta = dt$
- after integration: $t = C(\theta - \cos\theta\sin\theta)$
- we have a "big crunch" for θ=π et t=πC

evolution of the universe III

3) $\kappa = -1 \rightarrow \quad \dot{a}^2 = \frac{C}{a} + 1$

- always $\dot{a} \geq 0$ and the universe expands forever
- the substitution $a = C \sinh^2 \chi$ leads to $2C \sinh(\chi)^2 d\chi = dt$
- after integration:

$$t = C(\cosh \chi \sinh \chi - \chi)$$



finally we set $\Omega_\Lambda = 1$

$$\Omega_r = \Omega_m = \Omega_\kappa = 0$$

$$\dot{a}^2 = \frac{1}{3}\Lambda a^2$$

$$\boxed{a(t) = \exp(Ht),} \qquad H = \sqrt{\frac{\Lambda}{3}}$$

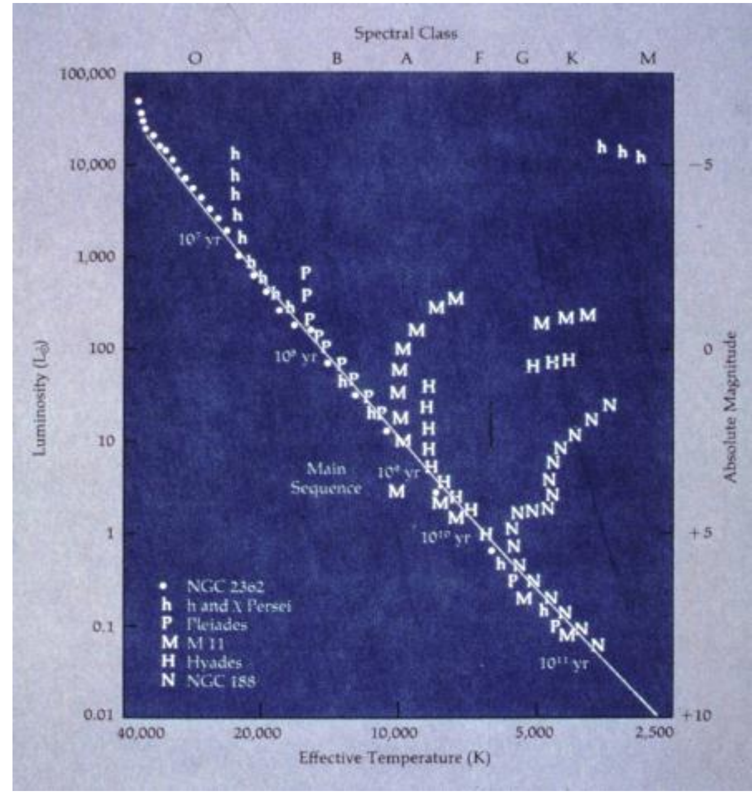
age of the universe revisited

we had: $$t_0 = \int_0^\infty \frac{dz}{H(z)(1+z)}$$

but for a matter-dominated universe:

$$H = H_0 \left(\frac{a}{a_0}\right)^{-3/2} = H_0(1+z)^{3/2}$$

$$H_0 t_0 = \int_0^\infty \frac{dz}{(1+z)^{5/2}} = \int_1^\infty \frac{du}{u^{5/2}} = -\frac{2}{3}\left.\frac{1}{u^{3/2}}\right|_1^\infty = \frac{2}{3}$$



$1/H_0$ ~ 9.8 Gyr/[H_0/100 km/s/Mpc] ~ 13.6 Gyr -> t_0 ~ 9 Gyr but oldest globular star clusters are older: 11-18 Gyr …??!!

summary

- **Methods**
  - FLRW metric for homogeneous and isotropic space
  - perfect fluid energy-momentum tensor
  - Einstein eq (Friedmann eq) and Bianchi identity
  - photon and particle geodesics
- **Results**
  - comoving coordinate system, redshift, Hubble law
  - expansion history of universe linked to contents
  - distances and redshift give constraints on expansion history and therefore on the contents

Resources (tiny subset!)

- Books & lecture notes
  - Scott Dodelson, “Modern Cosmology”, AP 2003
  - Ruth Durrer, “The Cosmic Microwave Background”, CUP 2008
  - Lots of reviews (e.g. Euclid theory group, arXiv:1206.1225)
  - Wayne Hu’s webpage, background.uchicago.edu
  - my (old) lecture notes, http://theory.physics.unige.ch/~kunz/lectures/cosmo_II_2005.pdf
- codes
  - Boltzmann codes: CAMB (camb.info), CLASS (class-code.net), etc
  - cosmoMC (with many likelihoods), cosmologist.info/cosmomc/
  - icosmo, icosmos, Fisher4Cast, etc
- lots of cosmological data sets are publicly available!
  - Planck: http://www.sciops.esa.int/index.php?project=planck&page=Planck_Legacy_Archive
  - WMAP (and others): Lambda archive, lambda.gsfc.nasa.gov
  - supernova data (e.g. supernova.lbl.gov/Union/) , …

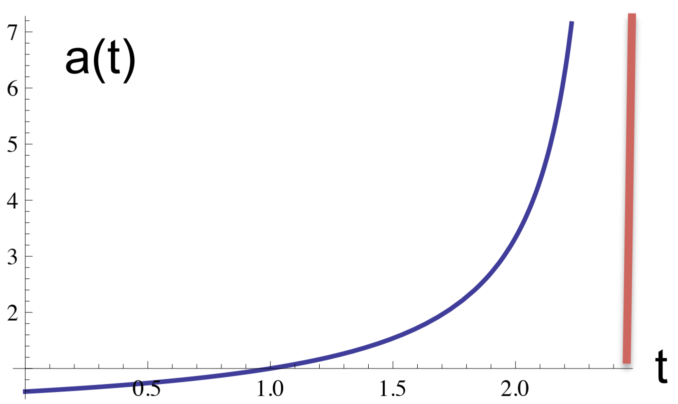
phantom DE and the big rip

- what happens if $p/\rho = w < -1$?
- (apart from classical and quantum instabilities)

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\Omega_m a^{-3} + (1-\Omega_m)a^{-3(1+w)}\right) \rightarrow H_0^2(1-\Omega_m)a^{-3(1+w)}$$

- solve for $a(t)$ with $a(t_0)=1$; let's pick $w = -5/3$ (similar for any $w<-1$)

$$\rightarrow a(t) = \left(1 - H_0\sqrt{1-\Omega_m}(t-t_0)\right)^{-1}$$



infinite expansion at finite age!
-> big rip

LTB and Backreaction

Two large classes of models:

- Inhomogeneous cosmology: Copernican Principle is wrong, Universe is not homogeneous (and we live in a special place).

- Backreaction: GR is a nonlinear theory, so averaging is non-trivial. The evolution of the 'averaged' FLRW case may not be the same as the average of the true Universe.

(& relativistic and large-angle effects)

Lemaitre-Tolman-Bondi

We live in the center of the world!

LTB metric: generalisation of FLRW to spherical symmetry, with new degrees of freedom

-> can choose a radial density profile, e.g. a huge void, to match one chosen quantity

☺ can mimic distance data (need to go out very far)

☺ demonstrates large effect from inhomogeneities

☹ unclear if all data can be mimicked (ISW, kSZ)

☹ mechanism to create such huge voids?

☹ fine-tuning to live in centre, ca $1:(1000)^3$ iirc

testing the geometry directly

Is it possible to test the geometry directly?

Yes! Clarkson et al (2008) -> in FLRW (integrate along ds=0):

$$H_0 D(z) = \frac{1}{\sqrt{-\Omega_k}} \sin\left(\sqrt{-\Omega_k}\int_0^z \frac{H_0}{H(u)} du\right)$$

$$\Rightarrow H_0 D'(z) = \frac{H_0}{H(z)} \cos\left(\sqrt{-\Omega_k}\int_0^z \frac{H_0}{H(u)} du\right)$$

$$\rightarrow \left(HD'\right)^2 - 1 = \sin^2(\cdots) = -\Omega_k \left(H_0 D\right)^2$$

It is possible to reconstruct the curvature by comparing a distance measurement (which depends on the geometry) with a radial measurement of H(z) without dependence on the geometry.

Backreaction

normal approach: separation into "background" and "perturbations"

$$g_{\mu\nu}(t,x) = \bar{g}_{\mu\nu}(t) + h_{\mu\nu}(t,x)$$

$$\rho(t,x) = \bar{\rho}(t) + \delta\rho(t,x)$$

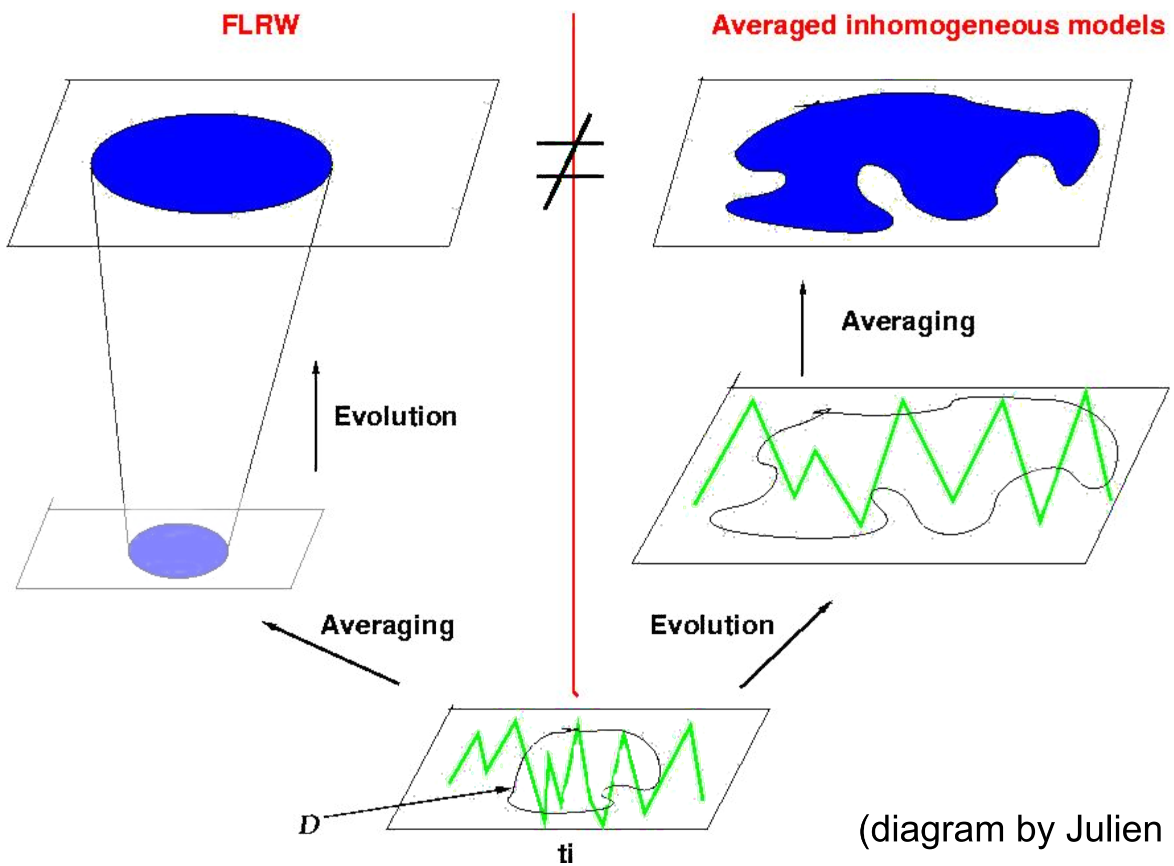
but which is the "correct" background, and why should it evolve as if it was a solution of Einsteins equations? The averaging required for the background does not commute with derivatives or quadratic expressions,

$$\partial_t \langle\phi\rangle \neq \langle\partial_t \phi\rangle \qquad \langle\theta^2\rangle \neq \langle\theta\rangle^2$$

-> can derive set of averaged equations, taking into account that some operations not not commute: "Buchert equations"

average and evolution

the average of the evolved universe is in general not the evolution of the averaged universe!



(diagram by Julien Larena)

Buchert equations

- Einstein eqs, irrotational dust, 3+1 split (as defined by freely-falling observers)
- averaging over spatial domain D
- $a_D \sim V_D^{1/3}$ [<-> enforce isotropic & homogen. coord. sys.]
- set of effective, averaged, local eqs.:

$$\frac{\dot{a}_D}{a_D} = \frac{8\pi G}{3}\langle\rho\rangle_D - \frac{1}{6}\left(\mathscr{Q} + \langle\mathscr{R}\rangle_D\right) \qquad 3\frac{\ddot{a}_D}{a_D} = -4\pi G\langle\rho\rangle_D + \mathscr{Q}$$

if this is positive then
it looks like dark energy!

$$\mathscr{Q} = \frac{2}{3}\left\langle\left(\theta - \langle\theta\rangle_D\right)^2\right\rangle_D - \langle\sigma_{ij}\sigma^{ij}\rangle_D$$

(θ expansion rate, σ shear, from expansion tensor Θ)

- $<\rho> \sim a^{-3}$
- looks like Friedmann eqs., but with extra contribution!

Backreaction

- ☺ is certainly present at some level
- ☺ could possibly explain (apparent) acceleration without dark energy or modifications of gravity
- ☺ then also solves coincidence problem

- ☹ amplitude unknown (too small? [*])
- ☹ scaling unknown (shear vs variance of expansion)
- ☹ link with observations difficult

[*] Poisson eq: $-\left(\frac{k}{Ha}\right)^2 \phi = \frac{3}{2}\delta$ (k = aH : horizon size)

=> Φ never becomes large, only δ ! (but this is not a sufficient argument)