Non-Gaussianity

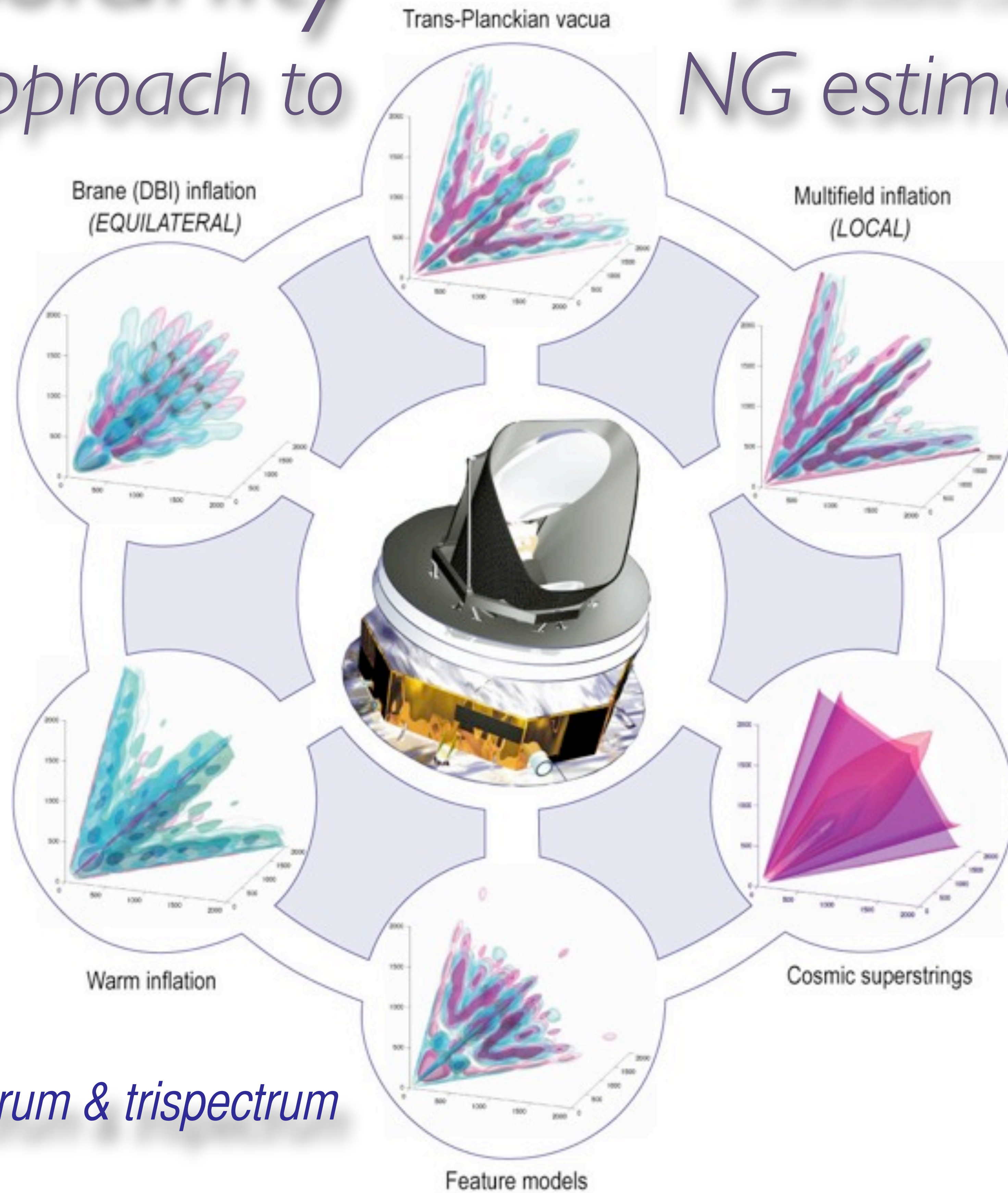
The Azores School on Observational Cosmology

Centro de Astrofísica da Universidade do Porto

General modal approach to NG estimation

James Fergusson & Paul Shellard

Centre for Theoretical Cosmology, DAMTP
University of Cambridge



LECTURE 2

- *Background: Primordial & CMB Bispectrum*
- *Importance of Separability*
- *Primordial mode expansions*
- *CMB modal estimation*
- *WMAP constraints on bispectrum & trispectrum*

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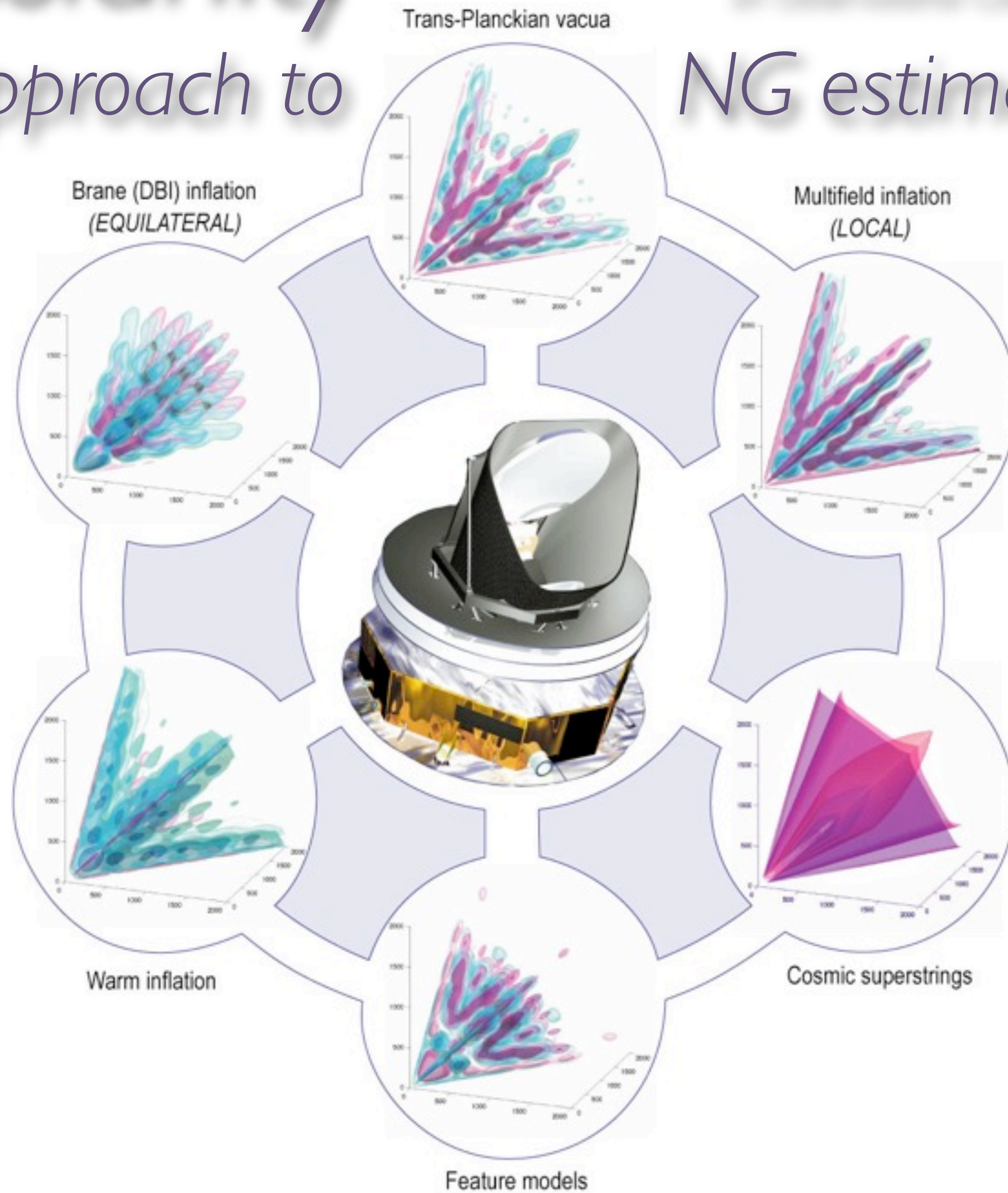
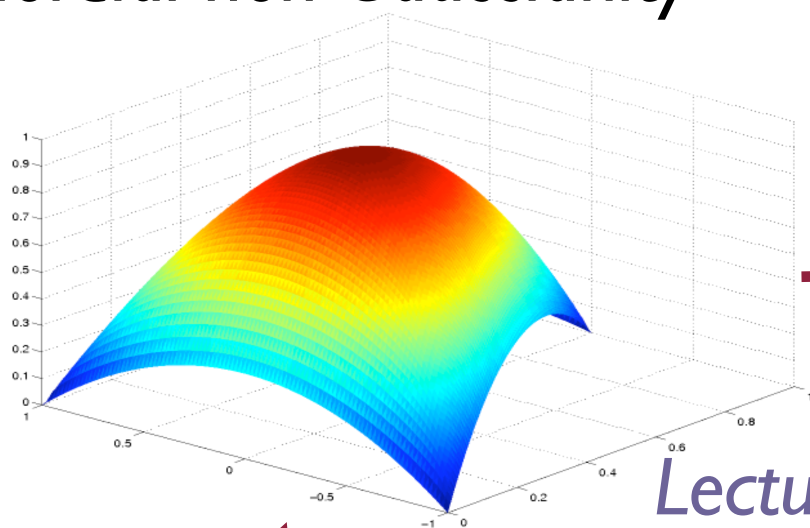


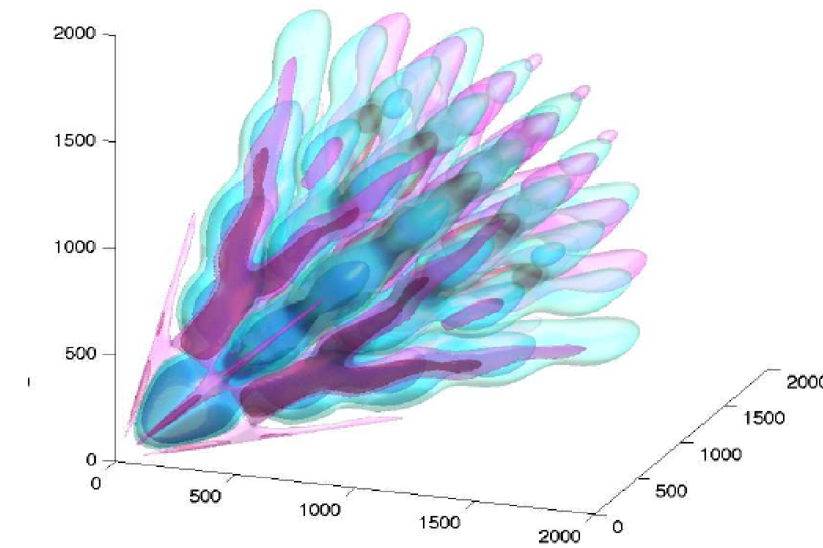
Overview

Primordial non-Gaussianity



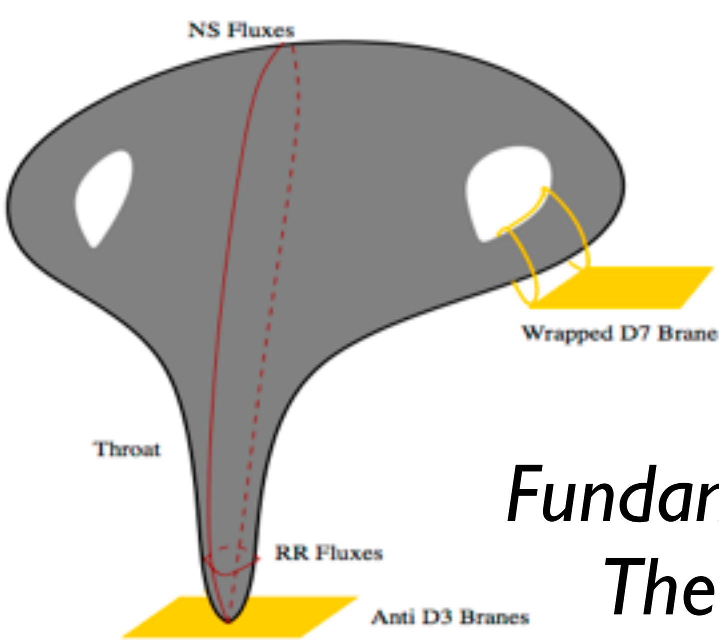
Lecture 1

CMB (or LSS) fingerprint

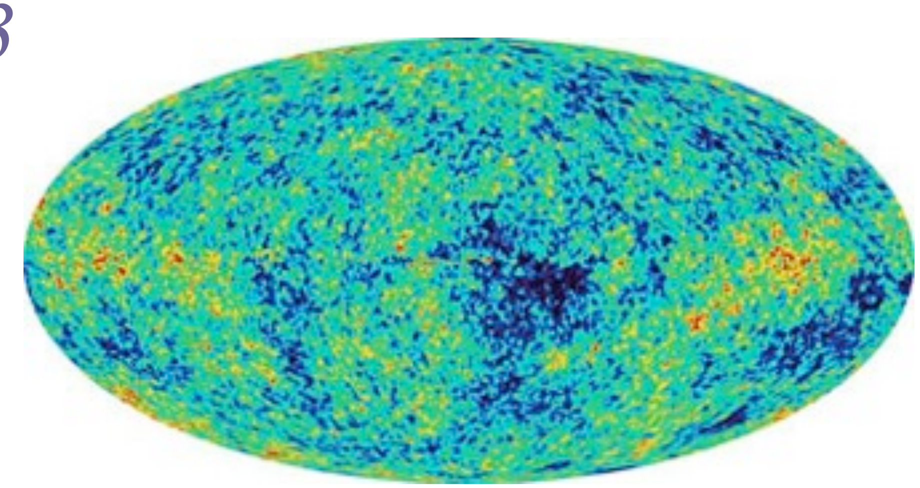


Lecture 2

Lecture 3



Fundamental Theory



Observational Data

BACKGROUND

- The primordial bispectrum and trispectrum* are defined by

$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3)\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3)B(k_1,k_2,k_3)$$
$$\langle \Phi(\mathbf{k}_1)\Phi(\mathbf{k}_2)\Phi(\mathbf{k}_3\Phi(\mathbf{k}_4)\rangle = (2\pi)^3\delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3+\mathbf{k}_4)T(k_1,k_2,k_3,k_4)$$

- For the CMB the bispectrum and trispectrum* are defined by

$$\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}\rangle = \left(\int d^2\hat{n}Y_{l_1m_1}(\hat{\mathbf{n}})Y_{l_2m_2}(\hat{\mathbf{n}})Y_{l_3m_3}(\hat{\mathbf{n}})\right)b_{l_1l_2l_3}$$
$$\langle a_{l_1m_1}a_{l_2m_2}a_{l_3m_3}a_{l_4m_4}\rangle = \left(\int d^2\hat{n}Y_{l_1m_1}(\hat{\mathbf{n}})Y_{l_2m_2}(\hat{\mathbf{n}})Y_{l_3m_3}(\hat{\mathbf{n}})Y_{l_4m_4}(\hat{\mathbf{n}})\right)t_{l_1l_2l_3l_4}$$

* Here we are considering for simplicity only diagonal free trispectra. In general isotropic trispectra depend on 6 parameters, (to uniquely define the quadrilateral) eg. 4 lengths and 2 angles. All statements we will make can be extended to general trispecra but my equations are long enough already.

BACKGROUND

- The two are related by a projection by transfer functions

$$< a_{l_1m_1} \ldots a_{l_pm_p} >= \int \frac{d^3k_1}{(2\pi)^3} \ldots \frac{d^3k_p}{(2\pi)^3} < \phi(\mathbf{k_1}) \ldots \phi(\mathbf{k_p}) > \Delta_{l_1}(k_1) \ldots \Delta_{l_p}(k_p) Y_{l_1m_1}(\hat{\mathbf{k}}_\mathbf{1}) \ldots Y_{l_pm_p}(\hat{\mathbf{k}}_\mathbf{p})$$

- The delta function in the primordial definition can be expanded as $\delta(\sum_1^p \mathbf{k}_i) = \int d^3x e^{i\mathbf{x}\cdot(\sum_1^p \mathbf{k}_i)}$

$$= 4\pi \sum_{l'_i m'_i} \left(\int x^2 dx j_{l'_1}(k_1x) \ldots j_{l'_p}(k_px) \right) \left(\int d^2\hat{\mathbf{x}} Y_{l'_1m'_1}(\hat{\mathbf{x}}) \ldots Y_{l'_pm'_p}(\hat{\mathbf{x}}) \right) Y_{l'_1m'_1}(\hat{\mathbf{k}}_\mathbf{1}) \ldots Y_{l'_pm'_p}(\hat{\mathbf{k}}_\mathbf{p})$$

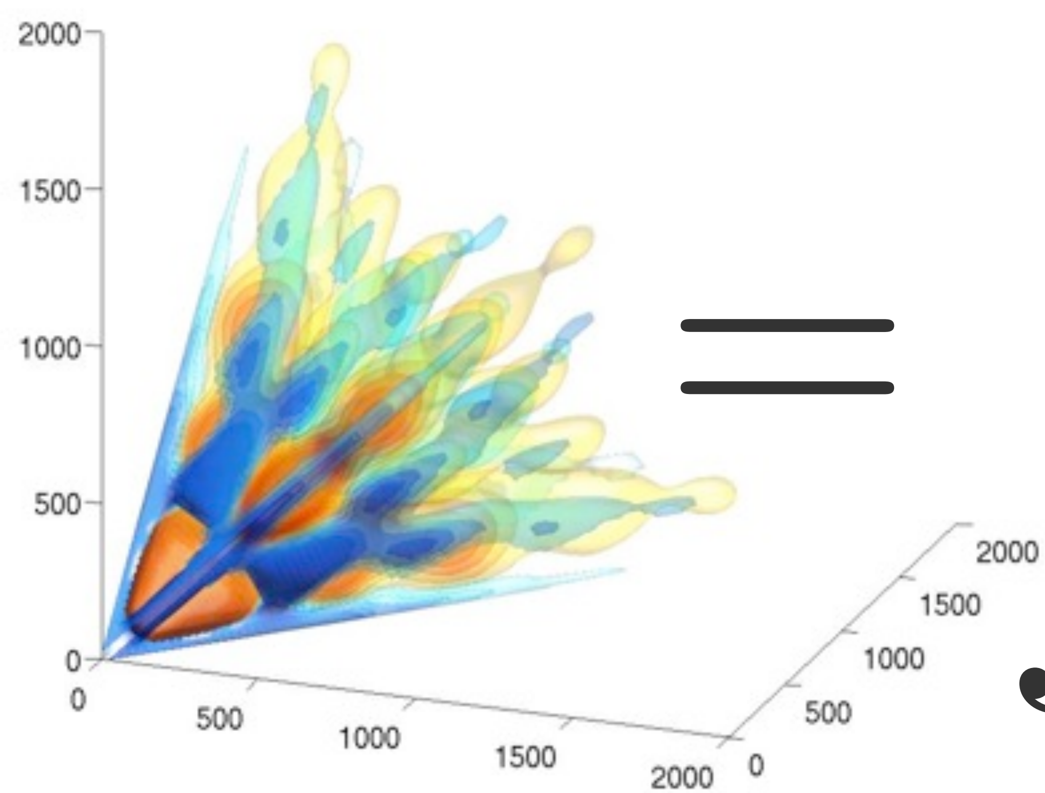
- The reduced quantities are then related by

$$b_{l_1l_2l_3} = \left(\frac{2}{\pi}\right)^3 \int x^2 dx \int dk_1 dk_2 dk_3 (k_1k_2k_3)^2 B(k_1,k_2,k_3) \Delta_{l_1}(k_1) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) j_{l_1}(xk_1) j_{l_2}(xk_2) j_{l_3}(xk_3)$$

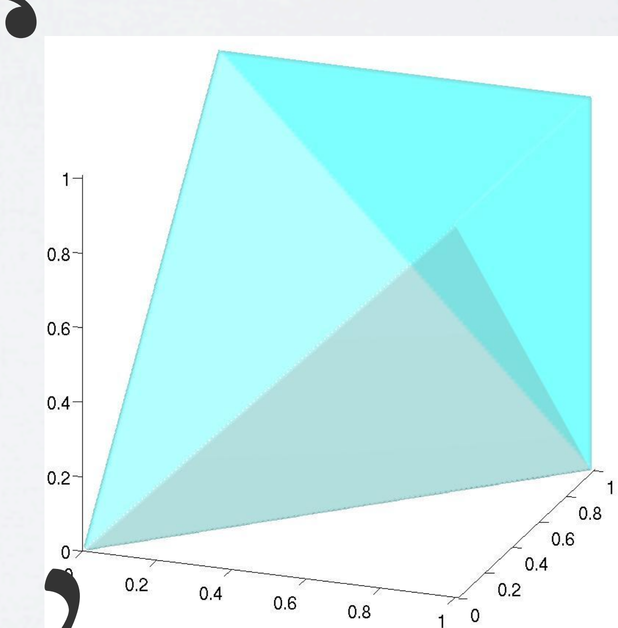
$$t_{l_1l_2l_3l_4} = \left(\frac{2}{\pi}\right)^4 \int x^2 dx \int dk_1 dk_2 dk_3 dk_4 (k_1k_2k_3k_4)^2 T(k_1,k_2,k_3,k_4) \Delta_{l_1}(k_1) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) \Delta_{l_4}(k_4) j_{l_1}(xk_1) \ldots.$$

BACKGROUND

$$b_{l_1 l_2 l_3} = \left(\frac{2}{\pi}\right)^3 \int_{\mathcal{V}_k} \left(k_1^2 k_2^2 k_3^2 B(k_1, k_2, k_3) \right) \times \left(\Delta_{l_1}(k_1)\Delta_{l_2}(k_2)\Delta_{l_3}(k_3) \int x^2 dx\, j_{l_1}(xk_1) j_{l_2}(xk_2) j_{l_3}(xk_3) \right)$$



$$= \int_{\mathcal{V}_k}$$



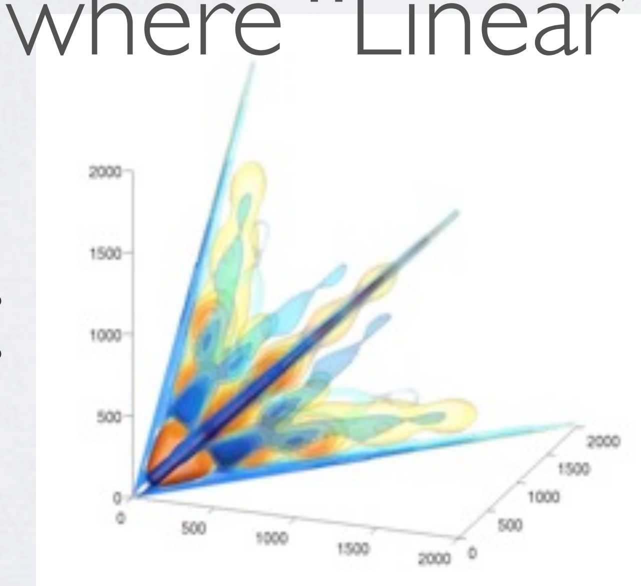
$$\times \Delta$$

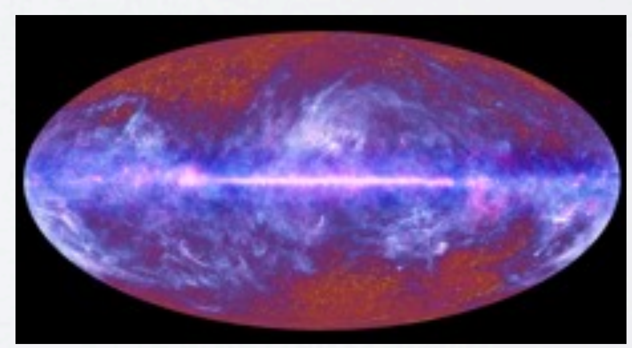
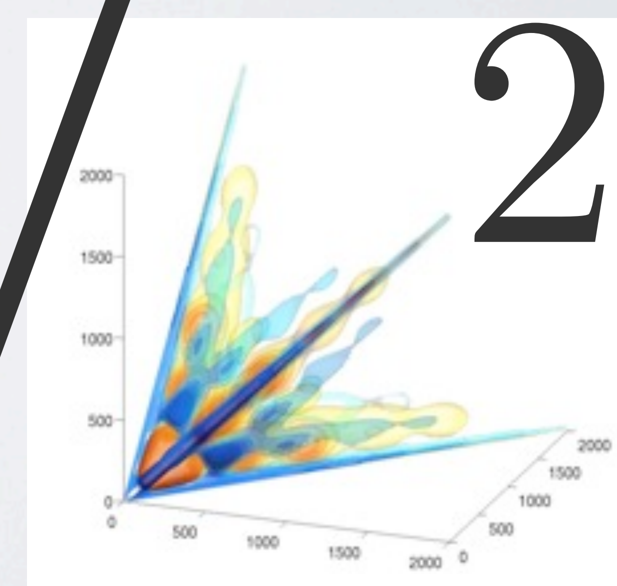
BACKGROUND

For a general polyspectrum the estimator takes the general form

$$\mathcal{E} = \sum_{l_i m_i l'_i m'_i} \frac{\left\langle a_{l_1 m_1} \ldots a_{l_p m_p} \right\rangle_{f_{NL}=1} C^{-1}_{l_1 m_1 l'_1 m'_1} \ldots C^{-1}_{l_p m_p l'_p m'_p} \left(a_{l'_1 m'_1} \ldots a_{l'_p m'_p} - \text{“Linear”} \right)}{\left\langle a_{l_1 m_1} \ldots a_{l_p m_p} \right\rangle_{f_{NL}=1} C^{-1}_{l_1 m_1 l'_1 m'_1} \ldots C^{-1}_{l_p m_p l'_p m'_p} \left\langle a_{l'_1 m'_1} \ldots a_{l'_p m'_p} \right\rangle_{f_{NL}=1}}$$

where “Linear” will be explained later.

$\mathcal{E} =$ 

$\times$  $/$ 

^{2}

BACKGROUND

This is very very difficult to calculate in general as it is a sum over l^{2p} elements which are themselves difficult to calculate

$$\mathcal{E} = \sum_{l_i m_i l'_i m'_i} \frac{\left\langle a_{l_1 m_1} \dots a_{l_p m_p} \right\rangle_{f_{NL}=1} C^{-1}_{l_1 m_1 l'_1 m'_1} \dots C^{-1}_{l_p m_p l'_p m'_p} \left(a_{l'_1 m'_1} \dots a_{l'_p m'_p} - \text{“Linear”} \right)}{\left\langle a_{l_1 m_1} \dots a_{l_p m_p} \right\rangle_{f_{NL}=1} C^{-1}_{l_1 m_1 l'_1 m'_1} \dots C^{-1}_{l_p m_p l'_p m'_p} \left\langle a_{l'_1 m'_1} \dots a_{l'_p m'_p} \right\rangle_{f_{NL}=1}}$$

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} \rangle = \left(\int d^2\hat{n} Y_{l_1 m_1}(\hat{\mathbf{n}}) Y_{l_2 m_2}(\hat{\mathbf{n}}) Y_{l_3 m_3}(\hat{\mathbf{n}}) \right) b_{l_1 l_2 l_3}$$

$$\langle a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} a_{l_4 m_4} \rangle = \left(\int d^2\hat{n} Y_{l_1 m_1}(\hat{\mathbf{n}}) Y_{l_2 m_2}(\hat{\mathbf{n}}) Y_{l_3 m_3}(\hat{\mathbf{n}}) Y_{l_4 m_4}(\hat{\mathbf{n}}) \right) t_{l_1 l_2 l_3 l_4}$$

BACKGROUND

- The only quantity that connects different l in the estimator is the CMB polyspectrum. And the only reason they are connected is through the corresponding primordial polyspectrum. All other parts are functions of a single k or l

- If we could write the primordial bispectra as the product of functions of single k then all the equations simplify.

$$b_{l_1 l_2 l_3} = \left(\frac{2}{\pi}\right)^3 \int_{\mathcal{V}_k} \left(k_1^2 k_2^2 k_3^2 B(k_1, k_2, k_3) \right) \times \left(\Delta_{l_1}(k_1) \Delta_{l_2}(k_2) \Delta_{l_3}(k_3) \int x^2 dx j_{l_1}(xk_1) j_{l_2}(xk_2) j_{l_3}(xk_3) \right)$$

SEPARABILITY

- The result is compact expressions of which the hardest to evaluate is only 3D

$$B(k_1, k_2, k_3) = X(k_1)Y(k_2)Z(k_3) + 5\,\text{permutations}.$$

$$b_{l_1 l_2 l_3} = \int x^2 dx \tilde{X}_{l_1}(x)\tilde{Y}_{l_2}(x)\tilde{Z}_{l_3}(x) + 5 \text{ permutations}$$

$$\mathcal{E} = \frac{1}{N}\int d^3x M_X(\mathbf{x})M_Y(\mathbf{x})M_Z(\mathbf{x})$$

$$\tilde{X}_l(x) = \int k^2 dk X(k)\Delta_l(k)j_l(kx) \qquad M_X(\mathbf{x}) = \sum_{lm}\tilde{X}_l(x)Y_{lm}(\hat{\mathbf{x}})\sum_{l'm'}C^{-1}_{lml'm'}a_{l'm'}$$

LINEAR TERM

- The linear term for the bispectrum is

$$3 < a_{l_1 m_1} a_{l_2 m_2} > a_{l_3 m_3}$$

Including it, the estimator becomes

$$\mathcal{E} = \frac{1}{N} \int d^3x \, (M_X(\mathbf{x}) M_Y(\mathbf{x}) M_Z(\mathbf{x}) - 3 < M_X(\mathbf{x}) M_Y(\mathbf{x}) > M_Z(\mathbf{x}))$$

and rather than calculate the full covariance matrix we just need to calculate the average of the product map.

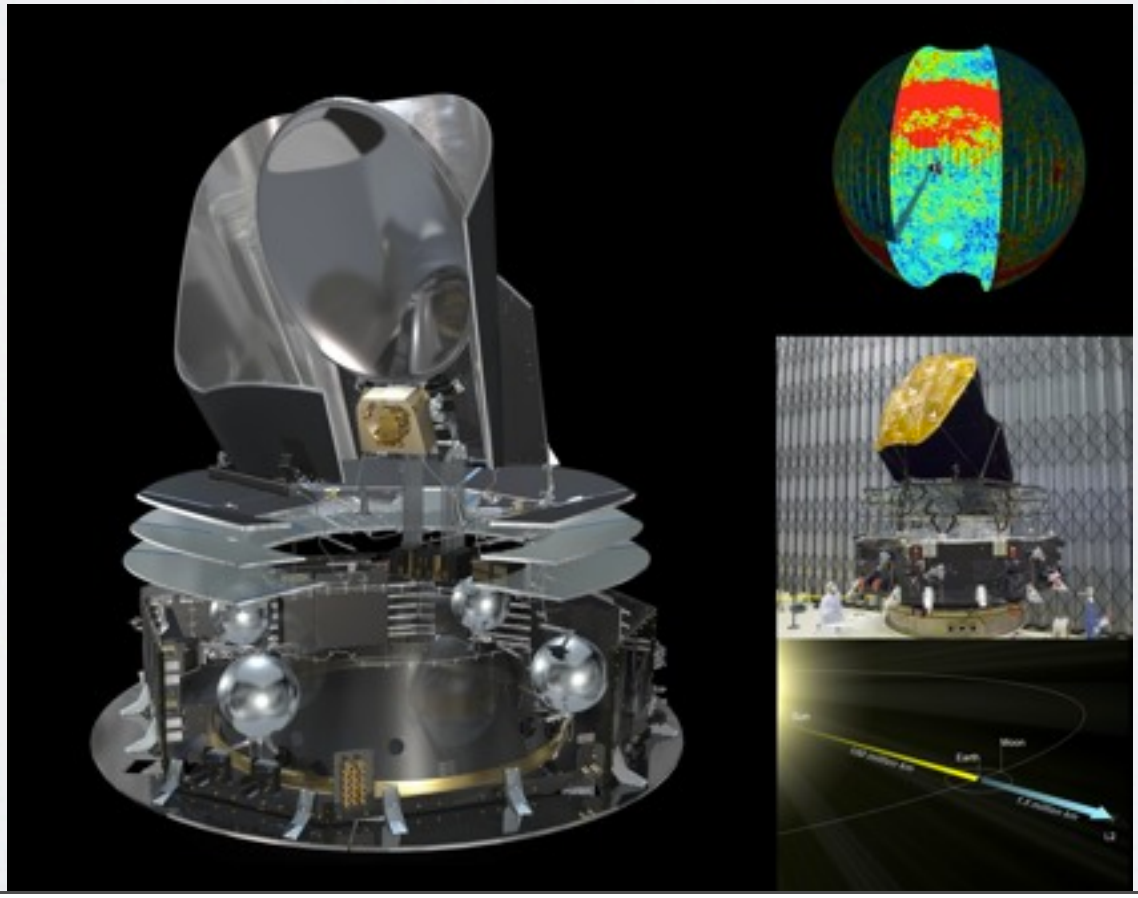
$$M_X(\mathbf{x}) = \sum_{lm} \tilde{X}_l(x) Y_{lm}(\hat{\mathbf{x}}) \sum_{l'm'} C^{-1}_{lml'm'} a_{l'm'}$$

EXPERIMENTAL EFFECTS?

- In a real experiment we must include the effect of beams noise and the mask

$$b_{l_1 l_2 l_3} \rightarrow f_{sky} b_{l_1} b_{l_2} b_{l_3} b_{l_1 l_2 l_3}$$

$$C_l \rightarrow f_{sky} \left(b_l^2 C_l + N_l \right)$$



SHAPE FUNCTION

- We wish to find a separable representation for the primordial bispectrum. As the bispectrum will be scale (or pseudo scale) invariant (ie $B(k,k,k) \propto k^{-6}$) it make sense to weight it before decomposition to flatten it out. Remembering

$$b_{l_1 l_2 l_3} = \left(\frac{2}{\pi}\right)^3 \int_{\mathcal{V}_k} \left(k_1^2 k_2^2 k_3^2 B(k_1,k_2,k_3)\right) \times \left(\Delta_{l_1}(k_1)\Delta_{l_2}(k_2)\Delta_{l_3}(k_3) \int x^2 dx j_{l_1}(xk_1) j_{l_2}(xk_2) j_{l_3}(xk_3)\right)$$

we see that we have a factor $(k_1 k_2 k_3)^2$ in front of the primordial bispectrum so we use it to divide out the scale defining a shape function:

$$S(k_1,k_2,k_3) = (k_1 k_2 k_3)^2 B(k_1,k_2,k_3)$$

ORTHONORMAL BASIS

What we would like is a basis which is both separable and orthonormal (for a suitable inner product) to expand the shape function in

$$S(k_1, k_2, k_3) = \sum_n \alpha_n R_n(k_1, k_2, k_3)$$

$$R_n(k_1, k_2, k_3) = r(k_1)r(k_2)r(k_3) + 5\,\mathrm{permutations}$$

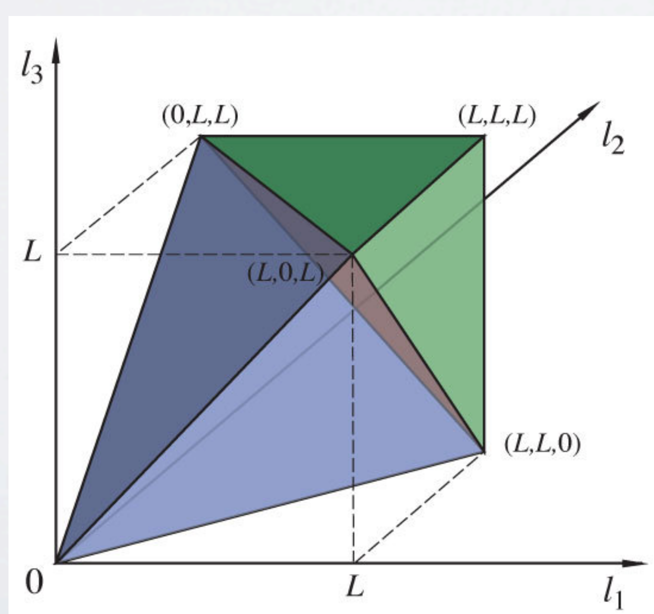
$$\langle R_n R_m \rangle = \delta_{nm}$$

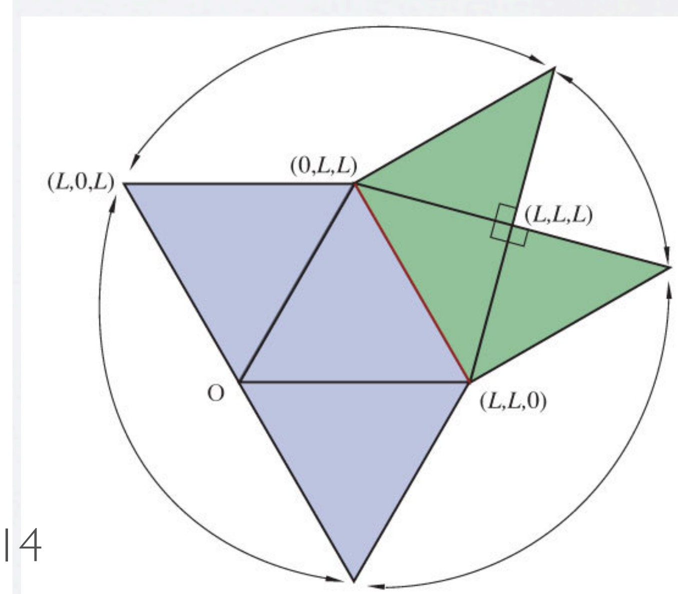
Then we could handle any model.

ORTHONORMAL BASIS

How to choose the inner product? Conservation of momentum requires the three k to obey the triangle condition and, as in the estimator we will be working to a particular maximum l, we will also restrict ourselves to a particular maximum k and choose our weight to be flat

$$\langle R_n R_m \rangle = \int_{\mathcal{V}} R_n R_m d\mathcal{V}$$





ORTHONORMAL BASIS

- Now how to construct our R?

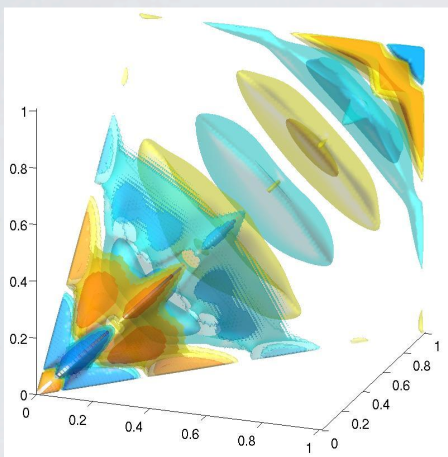
$$R_n(k_1,k_2,k_3) = \sum_m \lambda_{nm} Q_m(k_1,k_2,k_3)$$

$$Q_m(k_1,k_2,k_3) = \frac{1}{6}\left(q_i(k_1)q_j(k_2)q_k(k_3) + 5\,(permutations)\right)$$

Where the q are arbitrary functions and λ_{nm} is the product of some orthogonalisation procedure. We must also chose an ordering

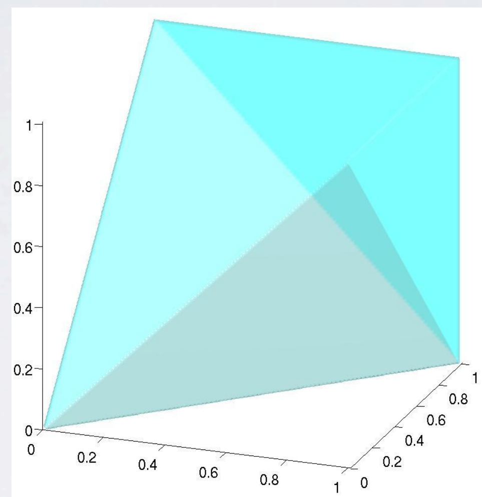
$\underline{0 \rightarrow 000}$	$4 \rightarrow 111$	$8 \rightarrow 022$	$12 \rightarrow 113$
$\underline{1 \rightarrow 001}$	$5 \rightarrow 012$	$9 \rightarrow 013$	$13 \rightarrow 023$
$2 \rightarrow 011$	$\underline{6 \rightarrow 003}$	$\underline{10 \rightarrow 004}$	$14 \rightarrow 014$
$\underline{3 \rightarrow 002}$	$7 \rightarrow 112$	$11 \rightarrow 122$	$\underline{15 \rightarrow 005}$ $\cdots$

ORTHONORMAL BASIS



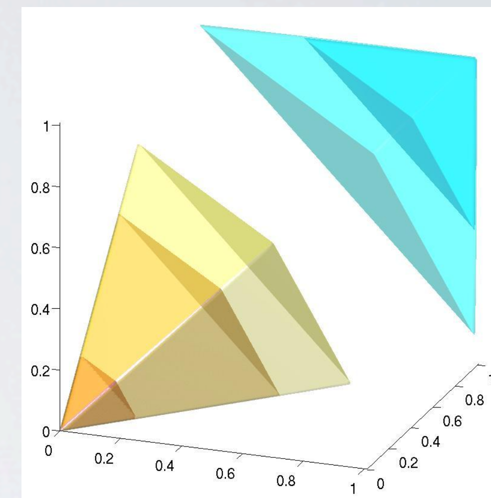
$S(k_1, k_2, k_3)$

$= \alpha_0$



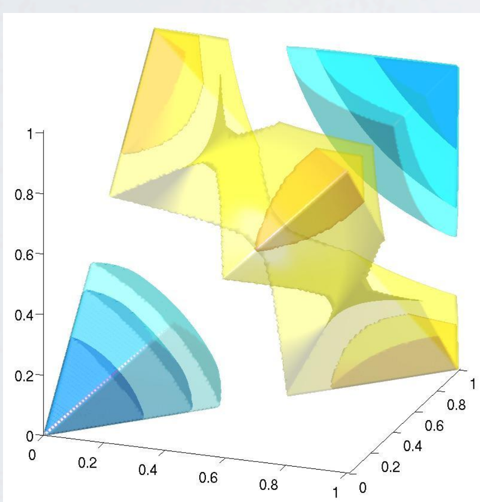
$000 \to 1$

$+\alpha_1$



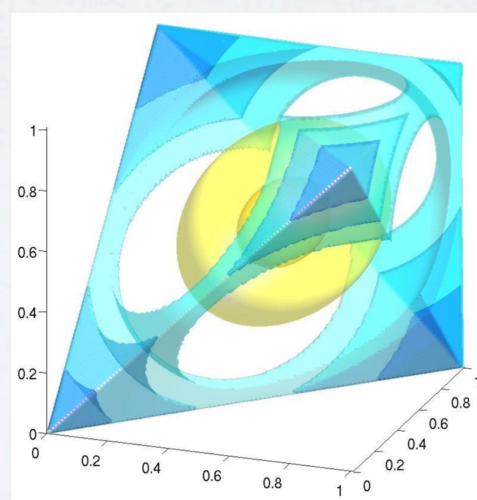
$001 \to k_1 + k_2 + k_3$

$+\alpha_2$



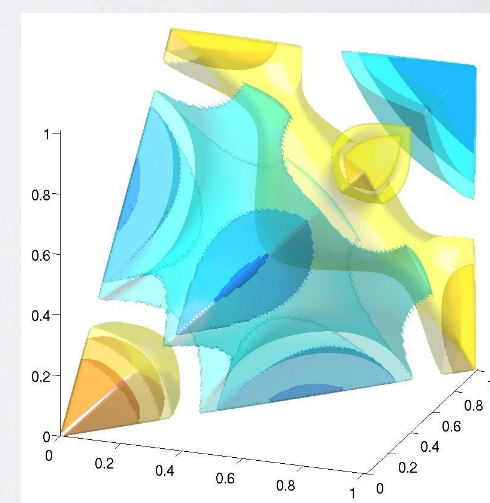
$011 \to k_1 k_2 + k_2 k_3 + k_3 k_1$

$+\alpha_3$



$002 \to k_1^2 + k_2^2 + k_3^2$

$+\alpha_4$



$111 \to k_1 k_2 k_3$

$\cdots$

ORTHONORMAL BASIS

Now we need to calculate λ_{nm}

$$\langle R_n R_m \rangle = \lambda_{nr} \lambda_{ms} \langle Q_r Q_s \rangle$$
$$\langle Q_r Q_s \rangle = \gamma_{rs}$$
$$I = \lambda \gamma \lambda^T$$

And rearranging, noting that λ_{nm} is lower triangular, we find it is the inverse of the Cholesky decomposition of the γ_{rs} matrix

$$\gamma = \lambda^{-1} {\lambda^{-1}}^T$$

ORTHONORMAL BASIS

Now we need to calculate the coefficients for the expansion

$$S = \sum_n \alpha_n^R R_n = \sum_n \alpha_n^Q Q_n$$

$$\alpha_n^R = \langle S R_n \rangle$$

$$\alpha_n^Q = \gamma_{nm}^{-1} \langle S Q_m \rangle$$

$$\alpha_n^R = {\lambda^{-1}}^T_{nm} \alpha_m^Q$$

ORTHONORMAL BASIS

We can now use this method to calculate the CMB bispectrum

$$b_{l_1 l_2 l_3} = \sum_n \alpha_n^Q \tilde{Q}^n_{l_1 l_2 l_3}$$

$$\tilde{Q}^n_{l_1 l_2 l_3} = \int x^2 dx \tilde{q}^{\{i}_{l_1}(x) \tilde{q}^{j}_{l_2}(x) \tilde{q}^{k\}}_{l_3}(x)$$

$$\tilde{q}^i_l(x) = \int dk\, q_i(k) \Delta_l(k) j_l(xk)$$

And estimator

$$\mathcal{E} = \frac{1}{N} \sum_n \alpha_n \beta_n$$

$$\beta_n^Q = \int d^3x M_i(\mathbf{x}) M_j(\mathbf{x}) M_k(\mathbf{x})$$

$$M_i(\mathbf{x}) = \sum_{lm} \tilde{q}^i_l(x) Y_{lm}(\hat{\mathbf{x}}) \sum_{l'm'} C^{-1}_{lml'm'} a_{l'm'}$$

WMAP EXAMPLES

If we consider the three models constrained by WMAP we find they can be represented by the following choices of monomials for the q and an ordering which only includes scale invariant combinations.

$$q_0(k) = k^{-1} \qquad 0 \rightarrow 003 \qquad \alpha^Q_{local} = \{2,\ 0,\ 0\}$$

$$q_1(k) = 1 \qquad 1 \rightarrow 012 \qquad \alpha^Q_{equi} = \{-1,\ 1,\ -2\}$$

$$q_2(k) = k \qquad 2 \rightarrow 111 \qquad \alpha^Q_{ortho} = \{-3,\ 3,\ -8\}$$

$$q_3(k) = k^2$$

The only difference is they never use orthonormality as they can read off the coefficients directly from their templates

WMAP EXAMPLES

There are limitations to this method. The first is by choosing monomials for q we can only use up to i=3 before the projection integral fails to converge

$$\tilde{q}_l^i(x) = \int dk\, q_i(k)\Delta_l(k) j_l(xk)$$

$$q_0(k) = k^{-1}$$

$$q_1(k) = 1$$

$$q_2(k) = k$$

$$q_3(k) = k^2$$

This is why it is much better to choose bounded functions eq. Legendre polynomials or Fourier modes as the q

Note: Due to the orthogonalisation procedure all polynomial choices lead to the same R. They only affect the stability of the method

WMAP EXAMPLES

The main problem with the primordial approach is that the projection from early to late time is in the "observational" β_n rather than the "theoretical" α_n. As you need to average over many maps to obtain error bars, and to calculate the linear term, this is very inefficient.

$$\mathcal{E} = \frac{1}{N}\sum_n \alpha_n \beta_n$$

$$\beta_n = \int d^3x M_i(\mathbf{x}) M_j(\mathbf{x}) M_k(\mathbf{x})$$

$$M_i(\mathbf{x}) = \sum_{lm} \tilde{q}_l^i(x) Y_{lm}(\hat{\mathbf{x}}) \sum_{l'm'} C^{-1}_{lml'm'} a_{l'm'}$$

NOTATION

What if we start instead decomposing the CMB bispectrum?

Starting with the estimator

$$\mathcal{E} = \sum_{l_i m_i l'_i m'_i} \frac{\left\langle a_{l_1 m_1} \dots a_{l_p m_p} \right\rangle_{f_{NL}=1} C^{-1}_{l_1 m_1 l'_1 m'_1} \dots C^{-1}_{l_p m_p l'_p m'_p} \left(a_{l'_1 m'_1} \dots a_{l'_p m'_p} - \text{“Linear”} \right)}{\left\langle a_{l_1 m_1} \dots a_{l_p m_p} \right\rangle_{f_{NL}=1} C^{-1}_{l_1 m_1 l'_1 m'_1} \dots C^{-1}_{l_p m_p l'_p m'_p} \left\langle a_{l'_1 m'_1} \dots a_{l'_p m'_p} \right\rangle_{f_{NL}=1}}$$

We can put this in a general form by defining

$$\langle \mathfrak{a}_{\wp} \rangle \equiv \langle a_{l_1 m_1} a_{l_2 m_2} \dots a_{l_p m_p} \rangle$$

$$\mathfrak{C}^{-1}_{\wp\wp'} \equiv C^{-1}_{l_1 m_1, l'_1 m'_1} \dots C^{-1}_{l_p m_p, l'_p m'_2 p}$$

Where $\wp$ represents the $\wp = \{l_1, m_1, l_2, m_2, \dots, l_p, m_p\}$ degrees of freedom

NOTATION

The estimator for a general polyspectrum is then defined as

$$\bar{\mathcal{E}} \equiv \frac{\sum_{\wp\,\wp'} \langle \mathfrak{a}_\wp \rangle \mathfrak{C}^{-1}_{\wp\wp'} \left(\mathfrak{a}_\wp - \mathfrak{a}^{lin}_\wp \right)}{\sum_{\wp\,\wp'} \langle \mathfrak{a}_\wp \rangle \mathfrak{C}^{-1}_{\wp\wp'} \langle \mathfrak{a}_\wp \rangle}$$

where $\mathfrak{a}^{lin}_\wp$ is the appropriate linear term

NOTATION

We will now go one step further by defining the weighted vectors (and matrix)

$$\mathcal{A}_\wp = \frac{\langle \mathfrak{a}_\wp \rangle}{\sqrt{C_{l_1} C_{l_2} ... C_{l_p}}}, \qquad \mathcal{B}_\wp = \frac{\mathfrak{a}_\wp - \mathfrak{a}_\wp^{lin}}{\sqrt{C_{l_1} C_{l_2} ... C_{l_p}}}, \qquad \mathcal{C}_{\wp\wp'} = \frac{\mathfrak{C}_{\wp\wp'}}{\sqrt{C_{l_1} C_{l'_1} ... C_{l_p} C_{l'_p}}},$$

And we can then write the estimator in matrix form as

$$\bar{\mathcal{E}} = \frac{\mathcal{A}^T \mathcal{C}^{-1} \mathcal{B}}{\mathcal{A}^T \mathcal{C}^{-1} \mathcal{A}}$$

CMB BASIS

If we then suppose the existence of an orthonormal basis at late time

$$\sum_{\wp} \bar{\mathcal{R}}_{n\wp}\bar{\mathcal{R}}_{n'\wp} = \delta_{nn'} \quad (\bar{\mathcal{R}}\bar{\mathcal{R}}^T = I)$$

again built from some separable functions $\bar{\mathcal{R}} = \bar{\lambda}\bar{\mathcal{Q}}$

$$\bar{\mathcal{R}}_{n\wp} = \frac{\int d^2n Y_{l_1m_1}(\hat{\mathbf{n}})Y_{l_2m_2}(\hat{\mathbf{n}})Y_{l_3m_3}(\hat{\mathbf{n}})}{v_{l_1}v_{l_2}v_{l_3}}\bar{R}_{n\,l_1l_2l_3}$$

$$\bar{R}_{nl_1l_2l_3} = \bar{\lambda}_{nm}\bar{Q}_{nl_1l_2l_3}(= q_iq_jq_k + 5 \text{ perms})$$

$$\sum_n \bar{\alpha}_n\bar{R}_{nl_1l_2l_3} = \frac{v_{l_1}v_{l_2}v_{l_3}b_{l_1l_2l_3}}{\sqrt{C_{l_1}C_{l_2}C_{l_3}}}$$

CMB BASIS

Then we can decompose our theory representing it as a set of modal coefficients

$$\mathcal{A}_{\wp} = \sum_n \alpha_n \mathcal{R}_{n\wp} \quad (\mathcal{A} = \mathcal{R}^T \alpha)$$

$$\alpha = \mathcal{R}\mathcal{A}$$

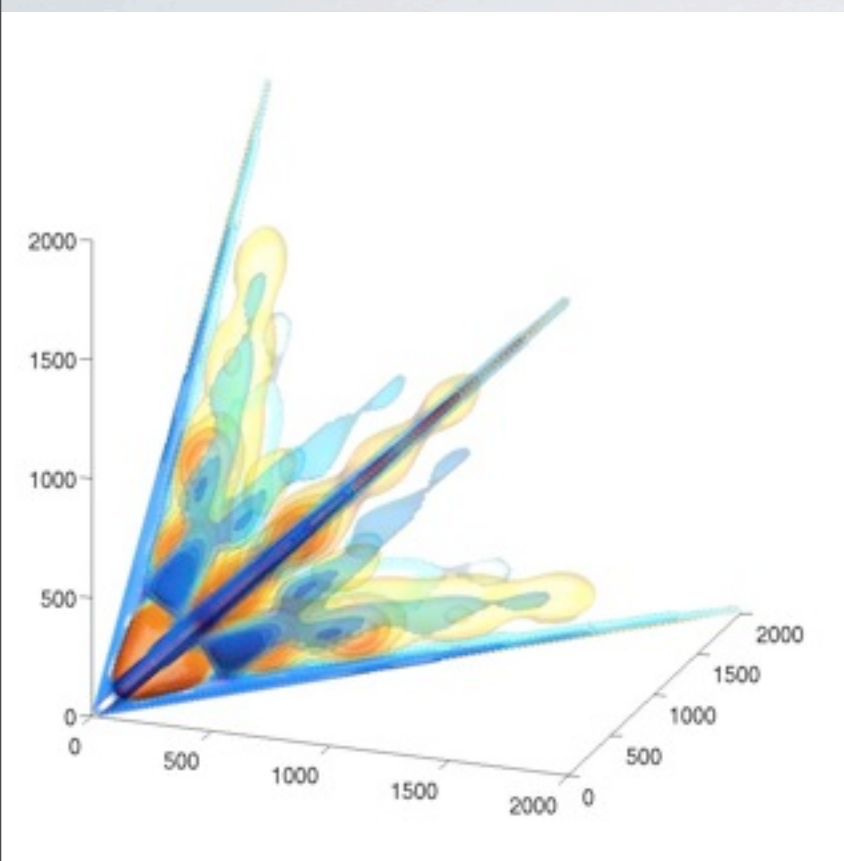
We will truncate our basis at some nmax so so we can also define a projection operator

$$\mathcal{P} = \mathcal{R}^T \mathcal{R}$$

And we take our theory to be completely described by this basis

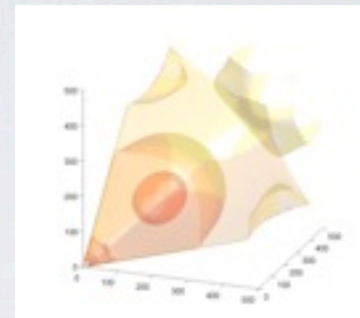
$$\mathcal{P}\mathcal{A} = \mathcal{A}$$

CMB BASIS

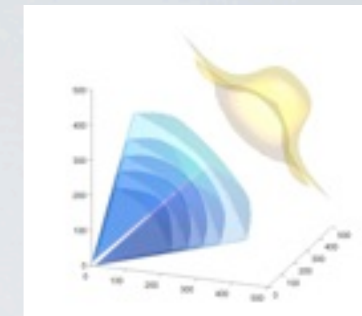


$=$

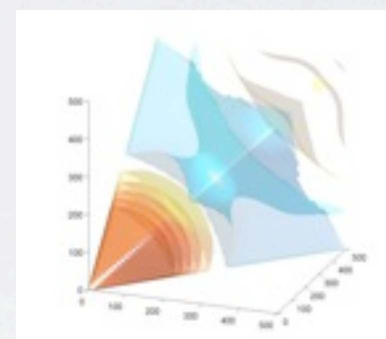
α_0



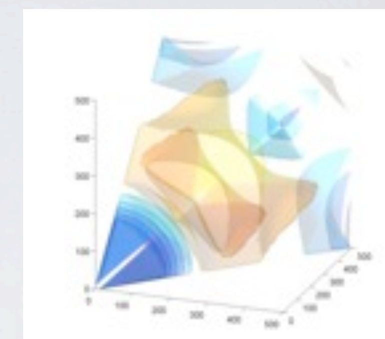
α_1



α_2

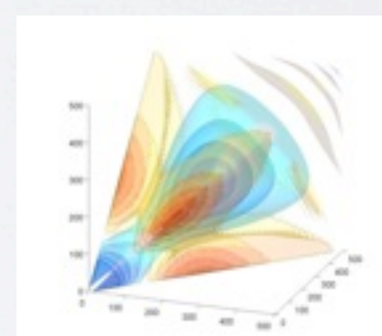


α_3



$+$

α_4



$\ldots\ldots$

CMB BASIS

We can perform the same modal decomposition on the data to obtain the estimator (we will assume the covariance is diagonal for now so $\mathcal{C} = I$)

$$\bar{\alpha} = \bar{\mathcal{R}}\mathcal{A} \rightarrow \mathcal{A} = \bar{\mathcal{R}}^T \bar{\alpha}$$

$$\bar{\beta} = \bar{\mathcal{R}}\mathcal{B} \rightarrow \mathcal{P}\mathcal{B} = \bar{\mathcal{R}}^T \bar{\beta}$$

$$\mathcal{E} = \frac{\sum \bar{\alpha}\bar{\beta}}{\sum \bar{\alpha}^2}$$

CMB EXAMPLES

Most late time methods can be written in this form. The only difference is orthonormality

For wavelets we chose the q to be the harmonic transform of the wavelet with differing sizes. They then build all combinations to form Q

For binned approaches the q are top hat functions for the relevant l ranges. Their combinations pick out individual sections of the bispectrum

All approaches are modal!

ORTHONORMAL BASIS

Now we have some nice properties. First

$$< \bar{\beta} >= \bar{\alpha}$$

the normalisation for the estimator is trivial

$$\mathcal{E} = \frac{\sum \bar{\alpha}\bar{\beta}}{\sum \bar{\alpha}^2}$$

and also all the projection is now in the calculation of alpha so the process is much more efficient*

* see the lecture of Casaponsa on Monday evening

ORTHONORMAL BASIS

And the covariance of β (which gives the variance of the estimator) reveals the importance of the linear term.

$$\langle\bar{\beta}_n\,\bar{\beta}_{n'}\rangle = \sum_{l_i m_i l'_i m'_i} \left\langle \left(\bar{\mathcal{R}}_{n l_1 l_2 l_3} \frac{a_{l_1 m_1} a_{l_2 m_2} a_{l_3 m_3} - 3\, C_{l_1 m_1, l_2 m_2} a_{l_3 m_3}}{\sqrt{C_{l_1} C_{l_2} C_{l_3}}} \right) \right.$$

$$\left. \times \left(\frac{a_{l'_1 m'_1} a_{l'_2 m'_2} a_{l'_3 m'_3} - 3\, C_{l'_1 m'_1, l'_2 m'_2} a_{l'_3 m'_3}}{\sqrt{C_{l'_1} C_{l'_2} C_{l'_3}}} \bar{\mathcal{R}}_{n l'_1 l'_2 l'_3} \right) \right\rangle$$

$$= \sum_{l_i m_i l'_i m'_i} \frac{\bar{\mathcal{R}}_{n l_1 l_2 l_3} \bar{\mathcal{R}}_{n' l'_1 l'_2 l'_3}}{\sqrt{C_{l_1} C_{l_2} C_{l_3} C_{l'_1} C_{l'_2} C_{l'_3}}} \left[6\, \langle a_{l_1 m_1} a_{l'_1 m'_1} \rangle \langle a_{l_2 m_2} a_{l'_2 m'_2} \rangle \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle \right.$$

$$+\ 9\, \langle a_{l_1 m_1} a_{l_2 m_2} \rangle \langle a_{l'_1 m'_1} a_{l'_2 m'_2} \rangle \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle - 9\, C_{l_1 m_1, l_2 m_2} \langle a_{l'_1 m'_1} a_{l'_2 m'_2} \rangle \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle$$

$$\left. -\ 9\, \langle a_{l_1 m_1} a_{l_2 m_2} \rangle C_{l'_1 m'_1, l'_2 m'_2} \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle + 9\, C_{l_1 m_1, l_2 m_2} C_{l'_1 m'_1, l'_2 m'_2} \langle a_{l_3 m_3} a_{l'_3 m'_3} \rangle + ... \right]$$

$$= 6 \sum_{l_i m_i l'_i m'_i} \bar{\mathcal{R}}_{n l_1 l_2 l_3} \frac{C_{l_1 m_1, l'_1 m'_1} C_{l_2 m_2, l'_2 m'_2} C_{l_3 m_3, l'_3 m'_3}}{\sqrt{C_{l_1} C_{l_2} C_{l_3} C_{l'_1} C_{l'_2} C_{l'_3}}} \bar{\mathcal{R}}_{n' l'_1 l'_2 l'_3}$$

$$= 6 \bar{\mathcal{R}} \mathcal{C} \bar{\mathcal{R}}^T = 6\mathrm{I}$$

ORTHONORMAL BASIS

If we also calculate the decomposition of the primordial basis modes projected forward

$$\bar{\mathcal{R}}\tilde{\mathcal{R}}^T = \Gamma \qquad \left(\tilde{\mathcal{R}}_l = \int_{\mathcal{V}_k} \mathcal{R}(k) \times \Delta\right)$$

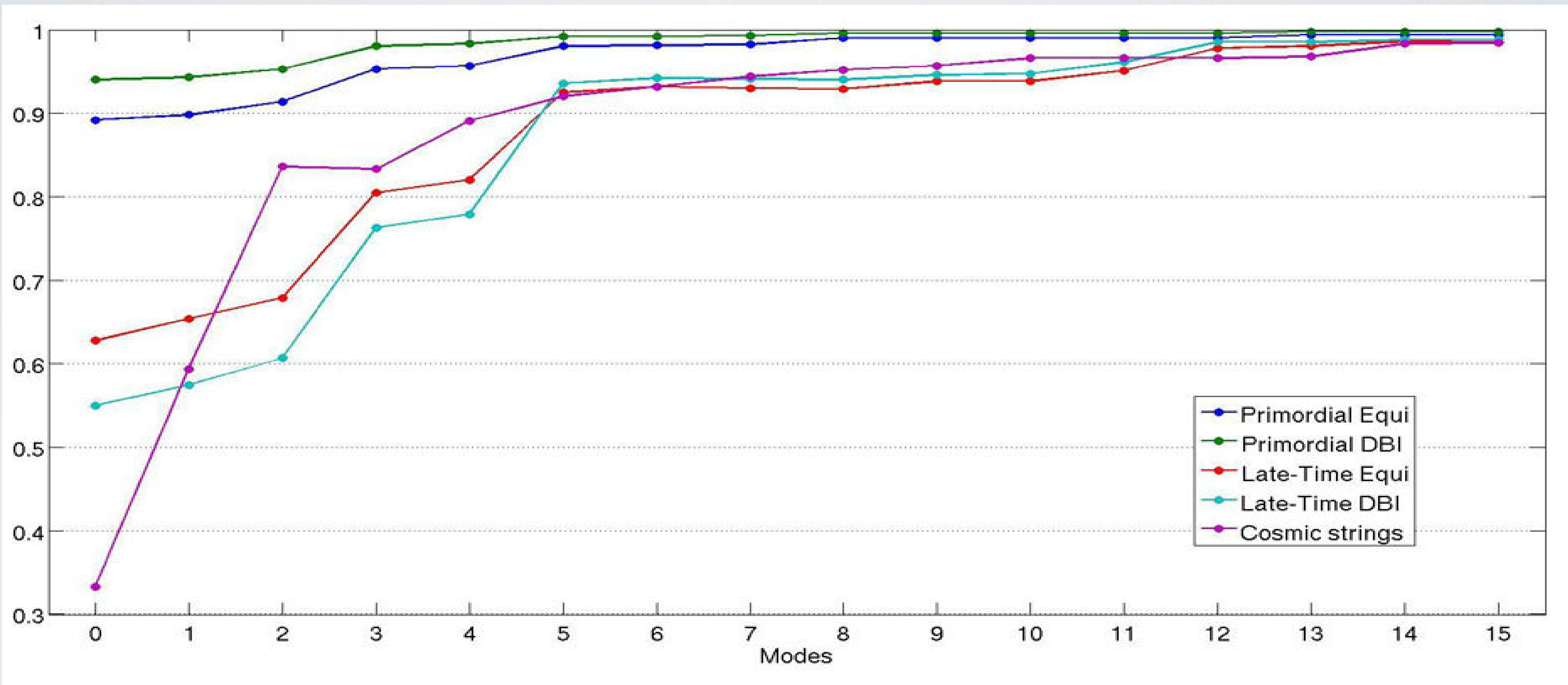
Then we can transform between the primordial and CMB expansions

$$\bar{\alpha}^{\mathcal{R}} = \Gamma\alpha^{\mathcal{R}}$$

$$\left(\bar{\alpha}^{\mathcal{Q}} = \bar{\lambda}\Gamma{\lambda^{-1}}^T\alpha^{\mathcal{Q}}\right)$$

CONVERGENCE

First, does it work?



Correlation between decomposition and original bispectra, both primordial and CMB

ESTIMATION

We have used these methods to constrain all scale invariant models......

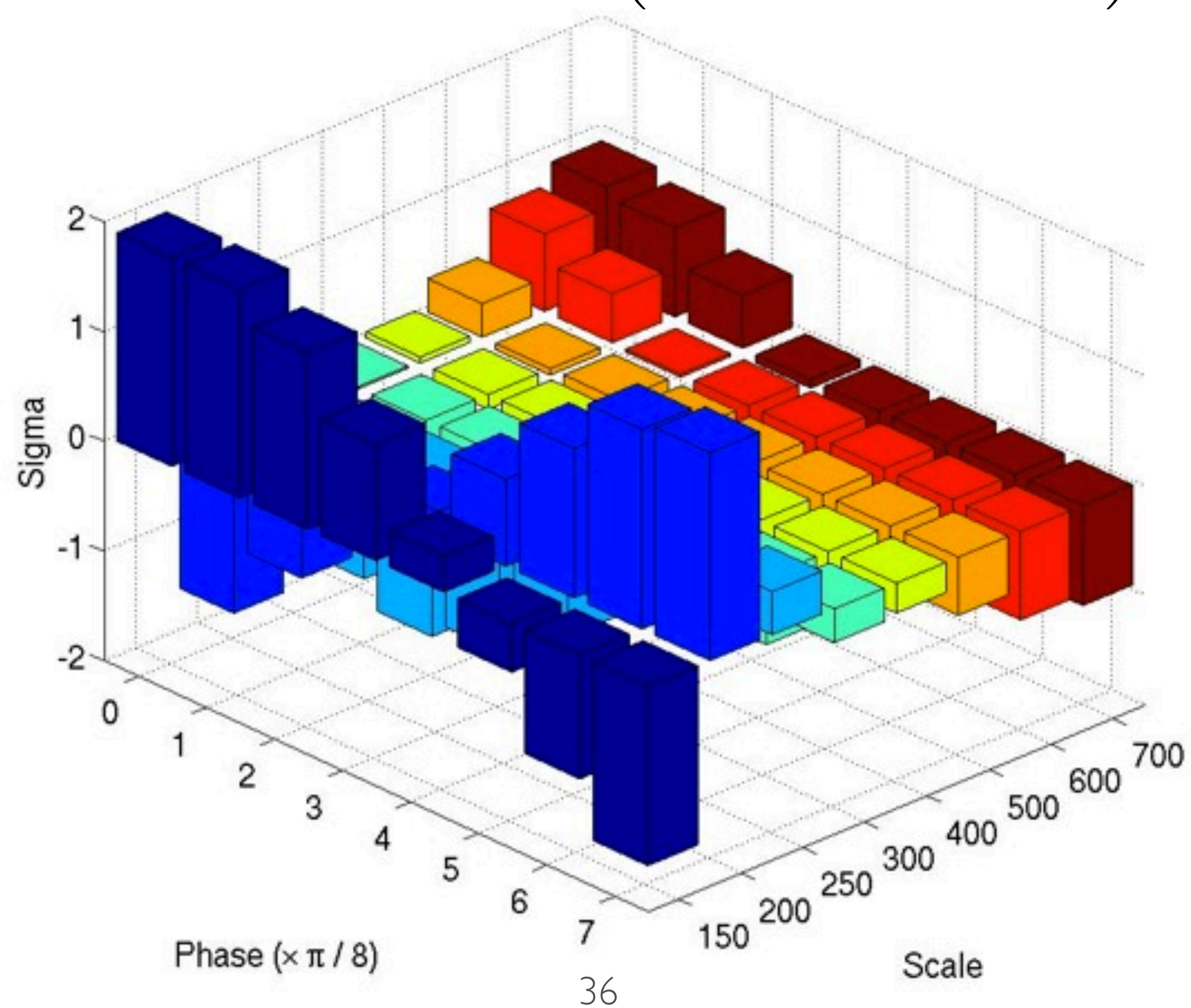
$$F_{NL} \rightarrow \sum \bar{\alpha}^2 = \sum \bar{\alpha}^2_{local}$$

Model	$F_{\rm NL}$	$(f_{\rm NL})$
Constant	35.1 ± 27.4	(149.4 ± 116.8)
DBI	26.7 ± 26.5	(146.0 ± 144.5)
Equilateral	25.1 ± 26.4	(143.5 ± 151.2)
Flat (Smoothed)	35.4 ± 29.2	(18.1 ± 14.9)
Ghost	22.0 ± 26.3	(138.7 ± 165.4)
Local	54.4 ± 29.4	(54.4 ± 29.4)
Orthogonal	-16.3 ± 27.3	(-79.4 ± 133.3)
Single	28.8 ± 26.6	(142.1 ± 131.3)
Warm	24.2 ± 27.3	(94.7 ± 106.8)

ESTIMATION

and an oscillatory model for a range of parameter space

$$S^{feat}(k_1, k_2, k_3) = \frac{1}{N} \sin\left(2\pi \frac{k_1 + k_2 + k_3}{3k^*} + \Phi\right)$$



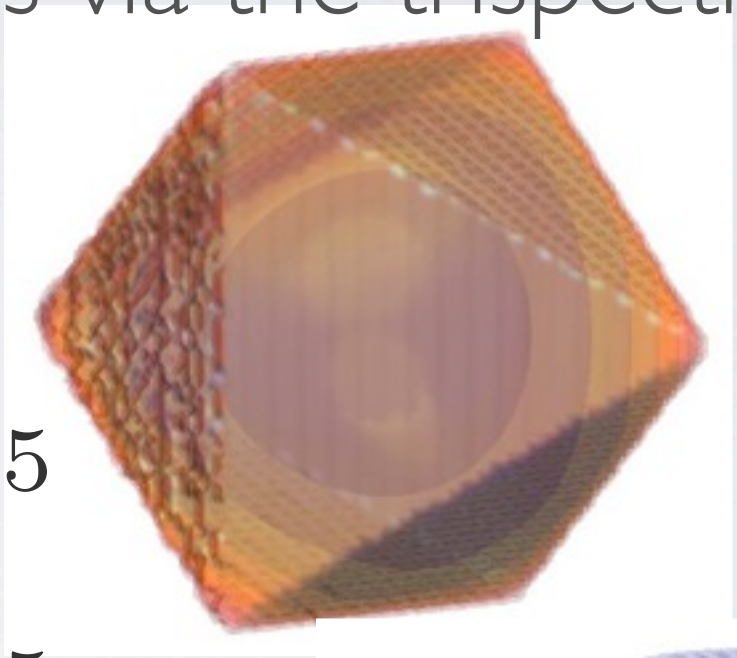
ESTIMATION

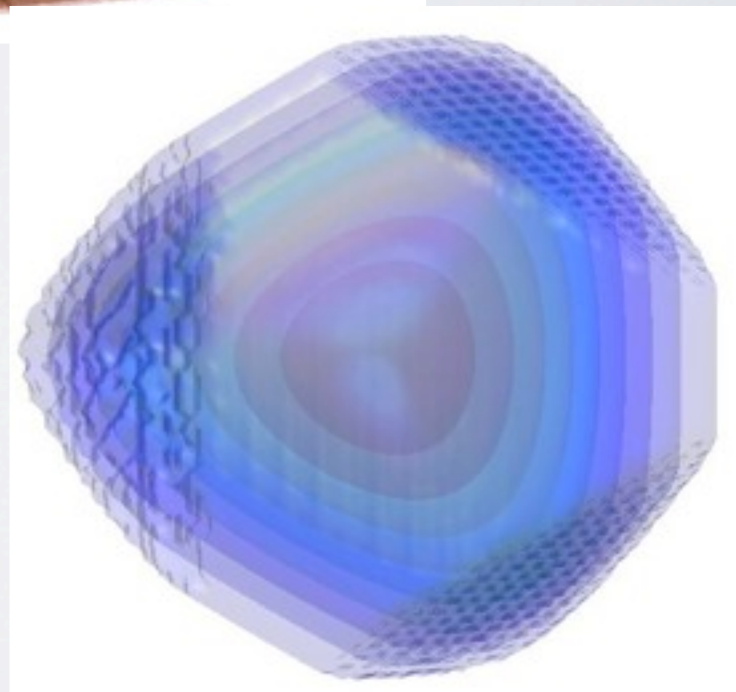
Once we have the α for each theory we can compute constrains for all of them simultaneously

$$\mathcal{E} = \frac{\sum \bar{\alpha}\bar{\beta}}{\sum \bar{\alpha}^2}$$

ESTIMATION

And a small selection of models via the trispectrum





$$G_{NL}^{local} = 1.62 \pm 6.98 \times 10^5$$

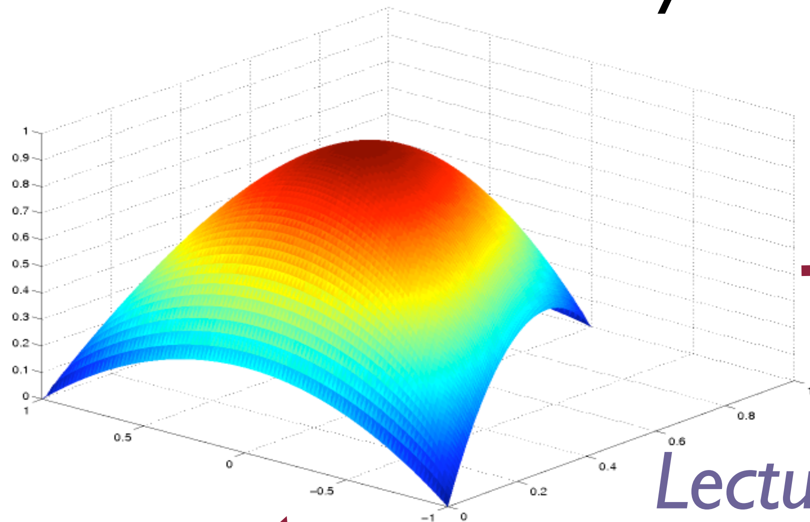
$$G_{NL}^{const} = -2.64 \pm 7.20 \times 10^5$$

$$G_{NL}^{equi} = -3.02 \pm 7.27 \times 10^5$$

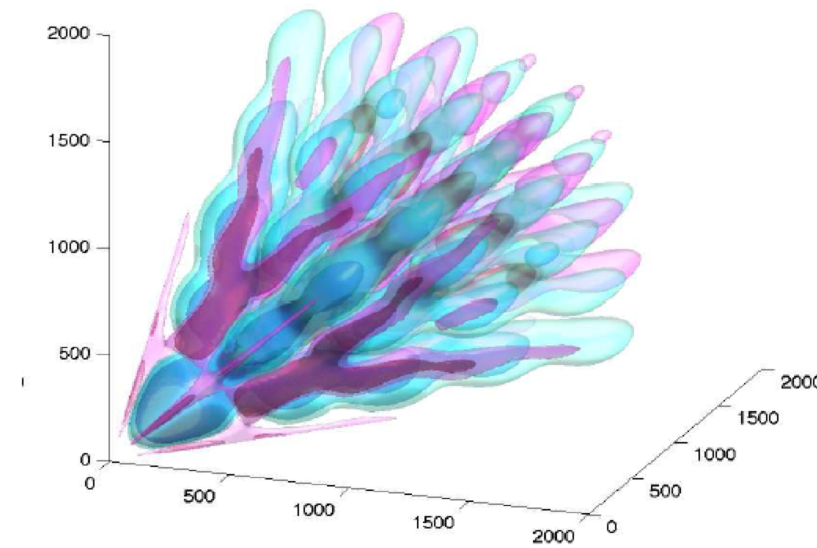
$$G\mu < 1.1 \times 10^{-6}$$

Overview

Primordial non-Gaussianity

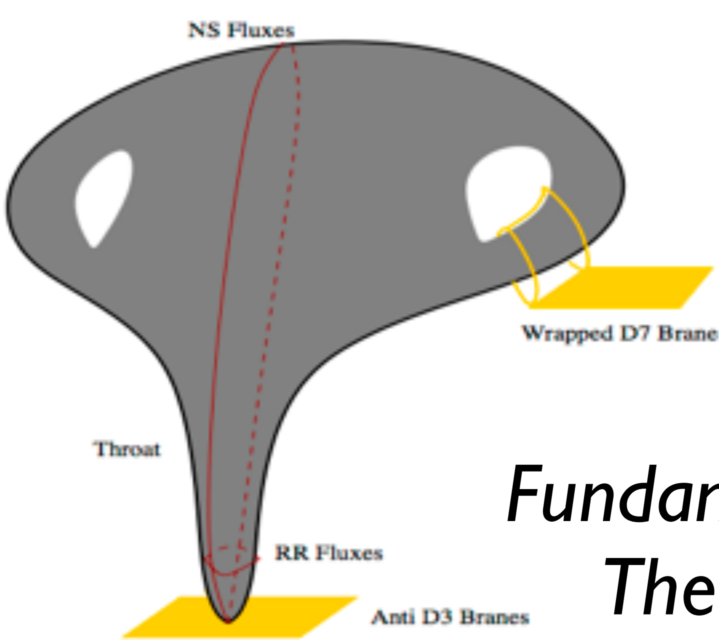


CMB (or LSS) fingerprint



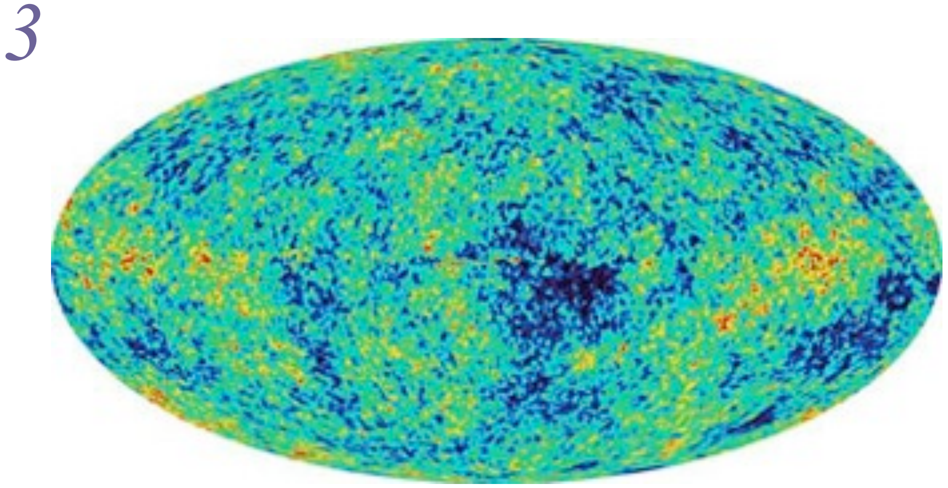
Lecture 1

Lecture 2



Lecture 3

Fundamental Theory



Observational Data