oscrox: stellar oscillation code

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Domain of applicability: adiabatic oscillations of spherical star: low ℓ modes

The oscillation equations

On linearising the equations of motion around the equilibrium model, taking all perturbations $\propto Y_{\ell m} e^{i\omega t}$ where $Y_{\ell m}$ are spherical harmonics and ω the angular frequency ($\omega = 2\pi\nu$), the equations governing small amplitude adiabatic oscillations can be expressed in the form (cf. Unno et al p.104)

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\xi\right) - \frac{g}{c^2}\xi + \left(1 - \frac{\ell(\ell+1)c^2}{\omega^2 r^2}\right)\frac{p'}{\rho c^2} - \frac{\ell(\ell+1)}{\omega^2 r^2}\Phi' = 0$$

$$\frac{1}{\rho}\frac{dp'}{dr} + \frac{g}{c^2}\frac{p'}{\rho} + \left(N^2 - \omega^2\right)\xi + \frac{d\Phi'}{dr} = 0$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi'}{dr}\right) - \frac{\ell(\ell+1)}{r^2}\Phi' - 4\pi G\rho\left(\frac{p'}{\rho c^2} + \frac{N^2}{g}\xi\right) = 0$$

where $g = \dfrac{GM_r}{r^2}, \quad c^2 = \Gamma_1\dfrac{P}{\rho}, \quad N^2 = \dfrac{g^2}{c^2}\left(1 - \Gamma_1\dfrac{d\log\rho}{d\log P}\right)$

and ξ, p', Φ' are the pertubations in radial displacement, Eulerian pressure and gravitational potential. g, c, ρ, P, Γ_1, N^2 are the acceleration due to gravity, sound speed, density, pressure, adiabatic exponent and Brunt-Väisäla frequency in the equilibrium model.

Following Vorontsov (OSC689) we define new variables

$$y_1 = \xi, \quad y_2 = \frac{p'}{\rho}, \quad y_3 = \Phi' \quad y_4 = \frac{d\phi'}{dr} - 4\pi G\rho\xi$$

and introduce dimensionless variables $x, \rho^*, c^*, n, g^*, \omega^*$ defined by

$$r = Rx, \quad \rho = \frac{M}{4\pi R^3}\rho^*, \quad c^2 = \frac{GM}{R}c^{*2}, \quad g = \frac{GM}{R^2}g^*, \quad N^2 = \frac{GM}{R^3}n^2, \quad \omega^2 = \frac{GM}{R^3}\omega^{*2}$$

On dropping the asterisks we obtain the equations in dimensionless form as

$$\frac{dy_1}{dx} = \left(\frac{g}{c^2} - \frac{2}{x}\right)y_1 + \left(\frac{\ell(\ell+1)}{x^2\omega^2} - \frac{1}{c^2}\right)y_2 - \frac{\ell(\ell+1)}{x^2\omega^2}y_3$$

$$\frac{dy_2}{dx} = \left(\omega^2 - n^2 + \rho\right)y_1 + \frac{n^2}{g}y_2 + y_4$$

$$\frac{dy_3}{dx} = \rho\, y_1 + y_4$$

$$\frac{dy_4}{dx} = -\frac{\ell(\ell+1)\rho}{x^2\omega^2}y_2 + \left(\frac{\ell(\ell+1)}{x^2}\right)\left(1 + \frac{\rho}{\omega^2}\right)y_3 - \frac{2}{x}y_4$$

Method of solution

The oscillations equations are a 4^{th} order homogeneous set which only have a solution for a discrete set of eigenvalues $\omega_{n\ell}$: the oscillation frequencies. The equations are solved using a shooting method with 4^{th} order Runge-Kutta integration. For a given value of ℓ and a value of the frequency ω, two independent solutions satisfying the surface boundary conditions (y_{i1}, y_{i2}) are integrated in to a matching point x_f. Likewise two independent solutions satisfying the conditions of regularity at the centre (y_{i3}, y_{i4}) are integrated out to x_f. If ω were an eigenfrequency these solutions would be continuous which requires that $D = det\{y_{ij}\} = 0$. In general this is not the case. Starting with a value of ω we increment ω until D changes sign, and then use Newton-Raphson to converge on the eigenvalue. We then increment ω again until D again changes sign and converge in on the next eigenvalue.

Input data from a stellar model

In principle the only data needed are the values of ρ, Γ_1 on a mesh of radius, r_i, and the surface value of the pressure; all other variables can be calculated from this data. In practice the code reads in $G, M, R, dLro2$ and the model variables

$$x_i = r_i/R, \quad q_i = M(r_i)/M \quad P_i, \quad \rho_i, \quad \Gamma_{1i}, \quad D_i \quad \text{on a mesh } i = 0 : N$$

where $$D(r) = \frac{1}{\Gamma_1} - \frac{d\log\rho}{d\log P}, \qquad dLro2 = \left(\frac{x^2}{\rho}\frac{d^2\rho}{dx^2}\right)_{x=0}$$

If $D, dLro2$ are not avaiable they are calculated in an auxilliary subroutine

Mid points

For 4^{th} order Runge Kutta we need the structure variables g, c^2, n^2, ρ at the mid points $(x_{i+1}+x_i)/2$. These are evaluated by interpolation on x^2, either simple linear interpolation or, if the model is smooth enough, cubic interpolation.

Expanding the mesh

The input mesh of a stellar model may be inadequate for calculating the frequencies since the mesh resolution may not be fine enough for there to be sufficient mesh points in a wavelength. In this case the mesh must be expanded. [*Note that this is true even if the coefficients in the oscillation equations are constants; for example to numerically solve the simple wave equation $y'' + \omega^2 y = 0$ one needs a mesh in x that contains several points within a wavelength see below.*]

The mesh is expanded by linear or cubic intepolation (in x^2), such that there are at least N_w mesh points within the shortest wavelength modes (high frequency $p-$ and low frequency $g - modes$) one seeks to calculate. Normally we take $N_w = 60$. The mesh may also be expanded for small x. The mesh can also be expanded or reduced by a constant factor as required.

Surface and Central boundary conditions

At the centre the solution must be regular and is developed as a power series in x. At the surface ($x = x_s$) the potential $y_3 = \phi'$ matches onto the corresponding solution of Laplace's equation, and either the Lagrangian pressure perturbation is set to zero ($y_2 = y_1/x^2$) or we use the isothermal reflective wave condition (Unno e al 1989, p.166)

$$y_2 - \frac{1}{x^2}y_1 - \left(\frac{\ell(\ell+1)}{\omega^2 x^3} - 4 - \omega^2 x^3\right)\frac{c^2}{\Gamma_1 x}y_1 + \left(\frac{\ell(\ell+1)}{\omega^2 x^3} - (\ell+1)\right)\frac{c^2 x}{\Gamma_1}y_3 = 0$$

Additional features

The code also has a pgplot subroutine that will plot the eigenmodes if required.

Accuracy and Mesh Resolution

Experiments with ESTA step 1

ESTA step 1

For step 1 in the comparison of frequency codes we were given a specific model of a main sequence star of mass $1.2M_\odot$ with initial abundance $X = 0.7, Z = 0.02$, evolved to a central hydrogen abundance $X_c = 0.690$

The model was given on a mesh of $N_0 = 902$ points with poor resolution in the central core the first point away from the core having $x = r/R = 0.0166263003$.

The frequencies of oscillation modes were calculated using the *oscrox* code for modes with $\ell = 0, 1, 2, 3$ in the range $100 \le \nu \le 4000\,\mu$Hz, for a variety of cases with different enhancements (and reductions) of the mesh resolution, and of the order of the integrator. *oscrox* in its standard form uses a 4^{th} order Runge-Kutta integrator, but the code was also run with a 2^{nd} order integrator iterated to convergence so that it solved the equations

$$\frac{dy_i}{dx} = f[x, y_k] \quad \text{in the form} \quad \frac{y_i(j+1) - y_i(j)}{x(j+1) - x(j)} = \frac{1}{2}\left(f[x(j+1), y_k(j+1)] + f[x(j), y_k(j)]\right)$$

The resuts for the following cases are shown in the figures.

2nd order integrator

The number of mesh points taken were $N = N_0 = 902$, the given mesh, and enhanced meshes enlarged by linear interpolation with $N = 2N_0, N = 4N0, N = 8N_0, N = 16N_0$. The results for $N = 8N_0 and N = 16N_0$ were the same. The same cases were run using cubic interpolation to enlarge the mesh but it did not make any significant difference to the values shown in Figure 1. This figure shows the difference in the frequencies $\delta\nu$ relative to the values with $N = 8 \times N_0$. The frequencies calculated on the input mesh $N_0 = 902$ are substantially different from the for large N, the differences being as large as 8μHz for the highest frequencies.

4th order integrator

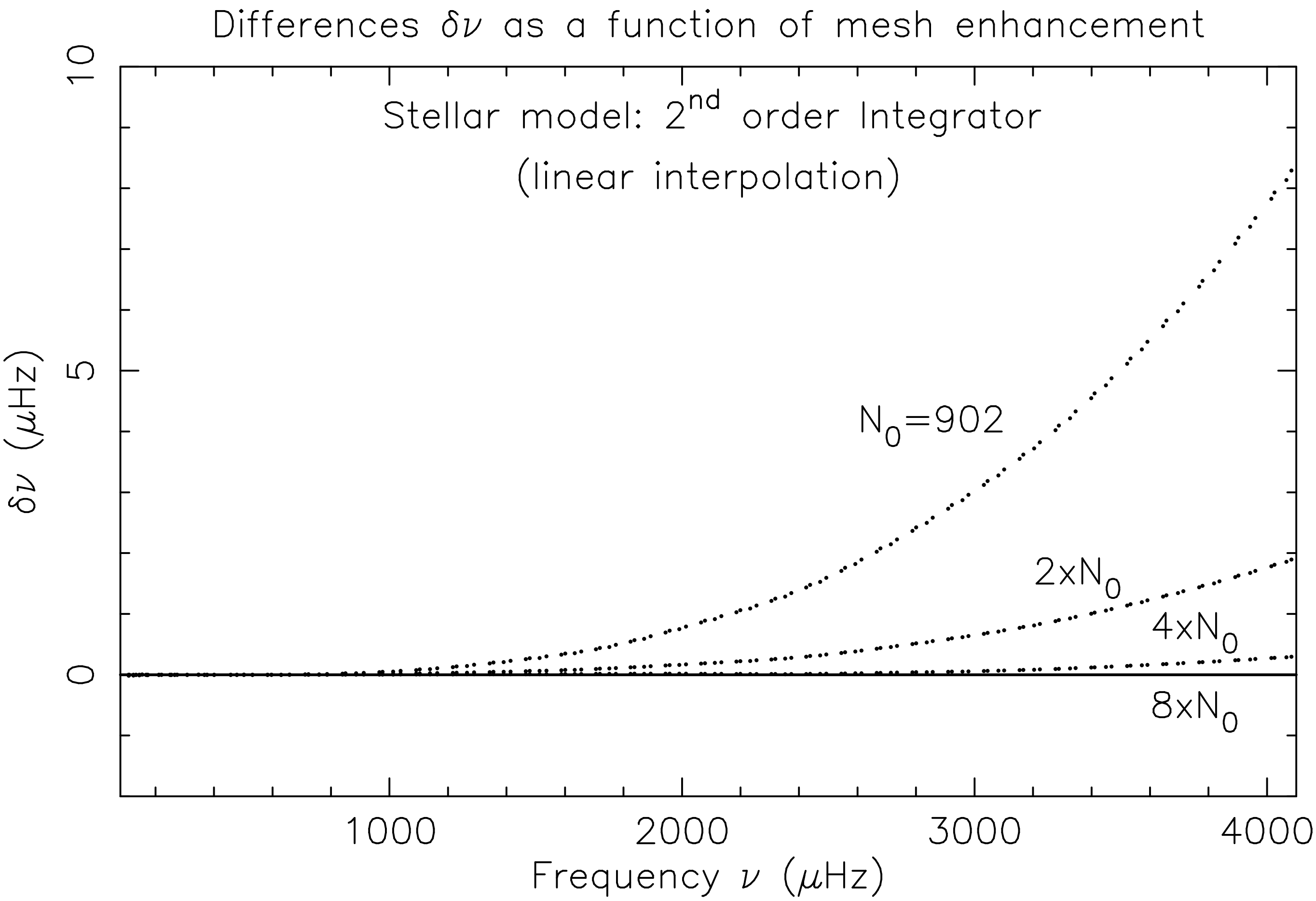
Since the 4th order Runge-Kutta integrator interpolates values at the mid points the basic model already has a doubling of mesh points. We therefore also calculated the model with $N = N_0/2 = 452$ mesh points, where we retained the 2nd mesh point $x = 0.0166263003$ and the surface mesh point at $x = 1.000718847$. Here we used cubic interpolation for both the mid points and mesh enlargement.

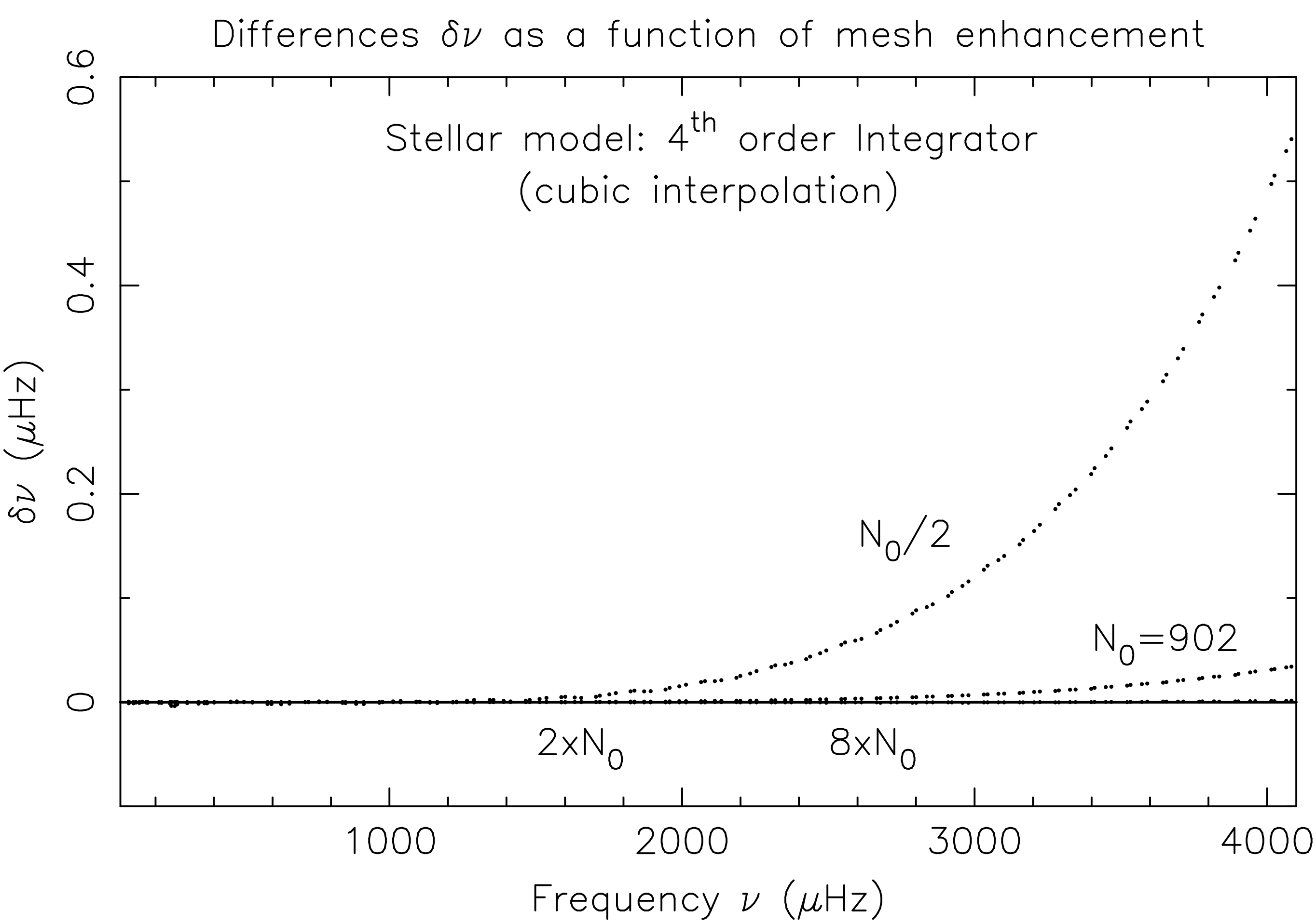
The results are shown in the second diagram which shows the diference between the frequencies calculated with $N = N_0 = 902$, $N = N_0/2 = 452$, $N = 2N_0$, and those with $N = 8N_0$. The differences are small, at high frequencies they range from 0.6μHz for $N = N_0/2$ to 0.03μHz for N_0, and the values for $N = 2N_0$ are indistinguishable from the values for $N = 8N_0$.

The calculations were repeated using linear interpolation rather than cubic to determine the mesh enlargemant and mid points. The same convergence with N is found in this case but the actual values at high frequencies differ by $\sim 0.1\mu$Hz.

Conclusion

These results demonstrate the importance of having sufficient mesh resolution in a wavelength, and of using an accurate integrator to solve the oscillation equations, to achieve an accuracy on the frequencies of 0.1μHz, which is the goal of COROT.


Differences δν as a function of mesh enhancement
Stellar model: 2nd order Integrator
(linear interpolation)
N0=902
2xN0
4xN0
8xN0
δν (μHz)
Frequency ν (μHz)
0
5
10
1000
2000
3000
4000


Differences δν as a function of mesh enhancement
Stellar model: 4th order Integrator
(cubic interpolation)
N0/2
N0=902
2xN0
8xN0
δν (μHz)
Frequency ν (μHz)
0
0.2
0.4
0.6
1000
2000
3000
4000

A simple example - Spherical Bessel Functions

To reinforce the above conclusions on the importance of mesh resolution and accuracy of the integrator, we here study the simple case of the non radial oscillations of a homogeneous compressible sphere of unit radius where, as shown by Rayleigh (1894), the equations governing the oscillations reduce to the spherical Bessel equation

$$\frac{d^2y}{dt^2} + \left(\omega^2 - \frac{\ell(\ell+1)}{t^2}\right) y = 0$$

For $\ell = 0$ this reduces to the harmonic equation with solution $y = \sin(\omega t)$ and, with $y = 0$ at the surface at $t = 1$, the eigenvalues $\omega_n = n\pi$. For $\ell = 1, 2, 3$ the eigenvalues can readily be determined from the known analytical forms of the spherical Bessel functions.

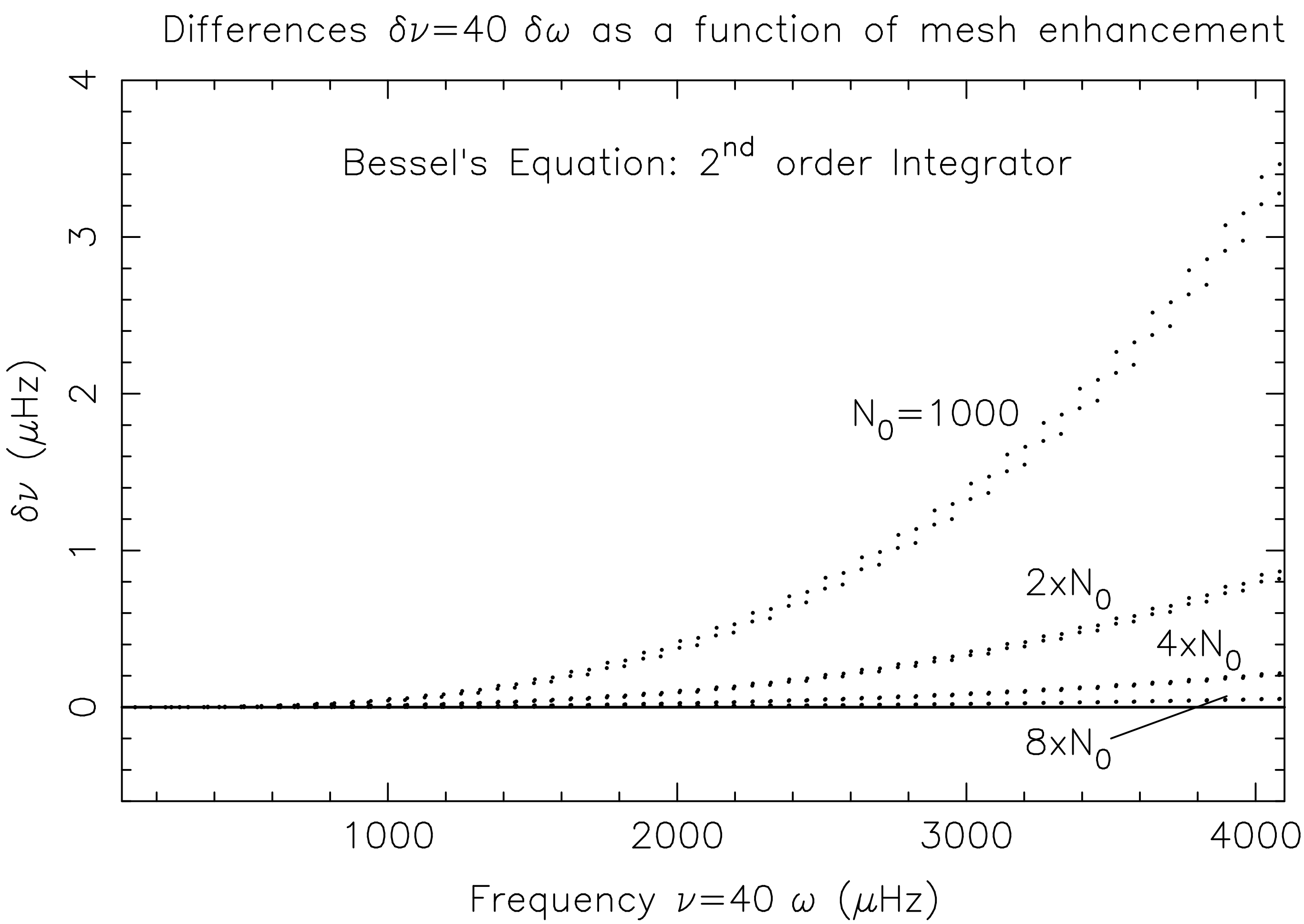
We repeat the experiment in the previous section, numerically integrating the Bessel equation using both 2^{nd} order and 4^{th} order integrators and compare the numerically determined eigenvalues with the known values for different mesh expansions. The resuts are shown in the following diagrams for modes up to $n = 32$ with an initial mesh resolution $N_0 = 1000$ on {t=0,1}. For convenience of comparison with the previous results we scale the all frequencies by a factor of $40 \approx \Delta/\pi$ where $\Delta \approx 123\mu$Hz is the large separation of frequencies of the stellar model considered in the previous section. Note that we see an ℓ dependence that was not visible in the results for the stellar model. This is due to the fact that for a real star the ℓ dependence is small in the outer layers (where c is small), whereas for the Bessel equation c is constant throughout the sphere so there is no reduction in the ℓ dependence in the outer layers.

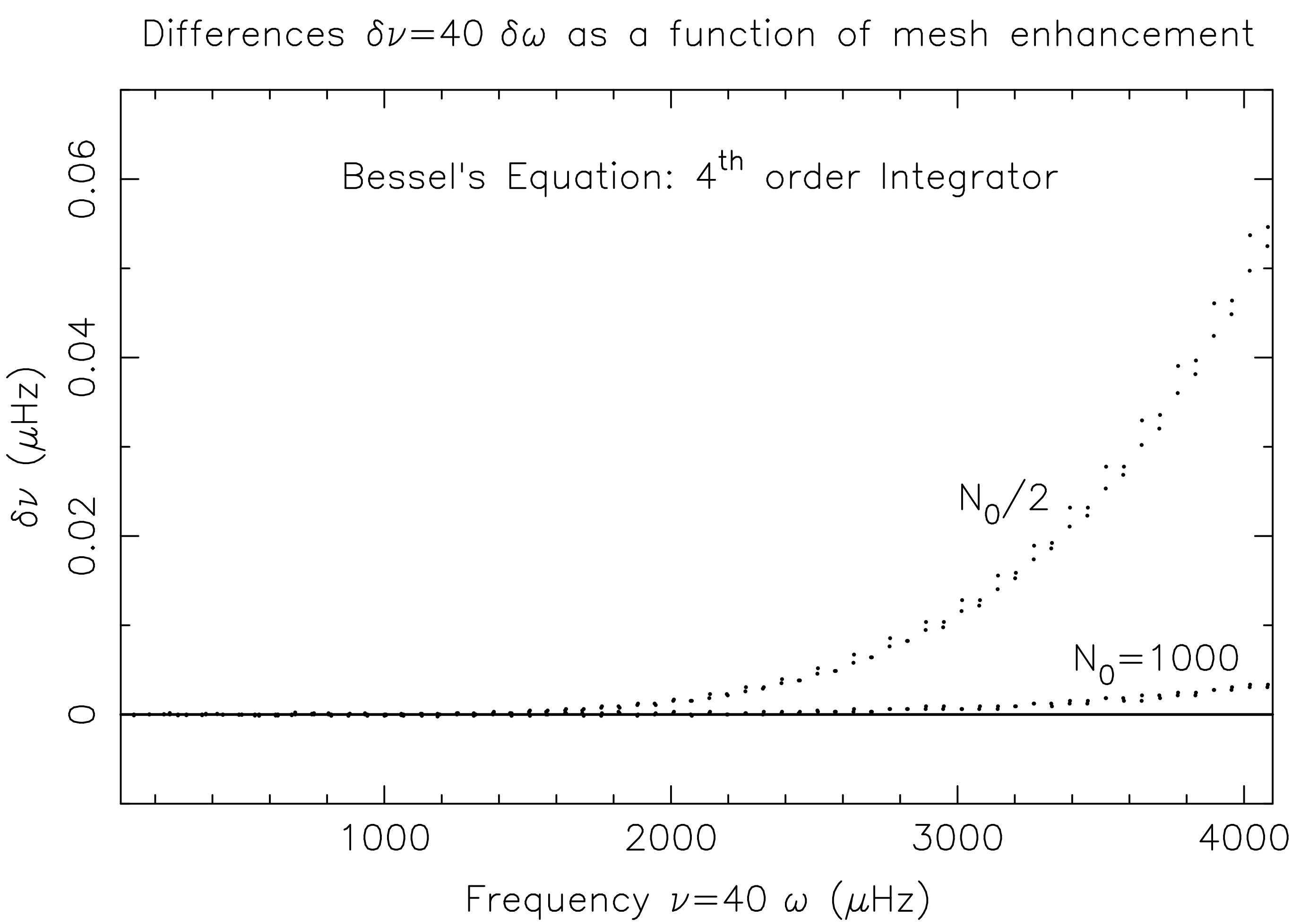
Again the 2^{nd} order integrator requires substantial expansion of the mesh to achieve an error in ω of 0.0025 which corresponds to 0.1μHz, the goal of COROT. The 4^{th} order integrator is very much more accurate - giving an error less than 0.00015 corresponding to 0.053μHz.

The accuracy in the baseline case, $N_0 = 1000$, is somewhat better for the Bessel equation than for the real star. This is because by choosing a mesh that is uniform in t we are making best use of the mesh points by having approximately the same number of mesh points in each wavelength, whereas in the stellar model the mesh points are not so favourably distributed.

This can be seen by solving the Bessel equation on the dimensionless acoustic radius of the stellar model and comparing the resulting eigenfrequencies with the values obtained with a uniform distribution of points. With the 2^{nd} order integrator on the mesh with $N_0 = 902$, the stellar mesh gives an eigenvalue for the mode $n = 32$ of 100.74, the uniform mesh gives 100.64, and the exact value is $32\pi = 100.531\ldots$. Scaling this up by the factor of 40 this corresponds to an error of 8.5μHz and 4.2μHz respectively.

On the other hand for the 4^{th} order integrator, with $N = 902$ the eigenvalues are 100.532 and 100.531 respectively, with errors corresponding to 0.03μHz and 0.005μHz. For $N = N_0/2$ the corresponding errors for the stellar mesh are 0.50μHz and 0.08μHz.


Differences δν=40 δω as a function of mesh enhancement
Bessel's Equation: 2nd order Integrator
N0=1000
2xN0
4xN0
8xN0
δν (μHz)
0
1
2
3
4
1000
2000
3000
4000
Frequency ν=40 ω (μHz)


Differences δν=40 δω as a function of mesh enhancement
Bessel's Equation: 4th order Integrator
N0/2
N0=1000
δν (μHz)
0
0.02
0.04
0.06
1000
2000
3000
4000
Frequency ν=40 ω (μHz)